

Introductory Control Systems

Mathematical Models of Physical Systems

- The *analysis* and *design* of *control systems* requires *quantitative mathematical models* of the physical systems to be controlled.
- A mathematical model may consist of *differential and/or algebraic equations*. The solution of these equations *describes* the *dynamics* of the system, that is, how the system responds to its expected input.
- A system may consist of a *single component*, or it may consist of *many different types of components* – mechanical, electrical, hydraulic, thermal, etc.
- A mathematical model may be *linear* or *nonlinear* depending on the system and the range of operation that is being modeled.
- If a system is nonlinear, it may be possible to *linearize* the model before applying linear analysis to the system. The extent to which this approach is applicable depends on the *strength* and *type* of *nonlinearities*.
- Mathematical models may be developed using *physical principles*. Using this approach, the analyst writes the differential and/or algebraic equations that are thought to describe the system dynamics.
- Laplace transforms are used to convert the *differential equations* into *transfer functions*.
- This approach is *limited* by the *analyst's ability* to:
 - *describe* the *physics* of the system (especially for complex systems)
 - *estimate* the important *parameters*

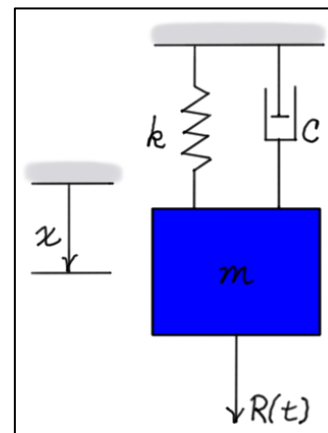
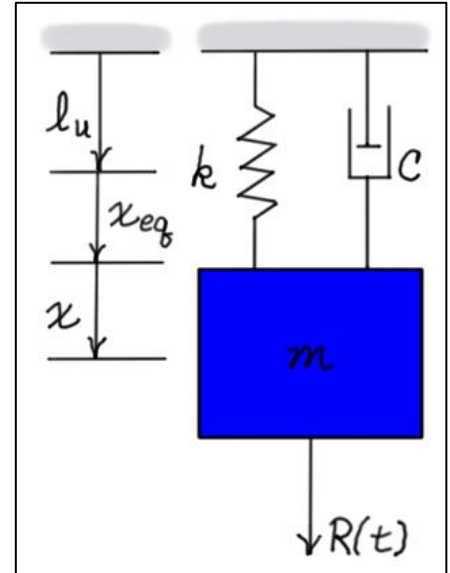


Fig. 1. Spring-Mass-Damper System

Example: Spring-Mass-Damper System

Nomenclature

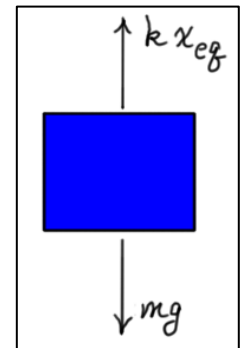
- m : mass of the block
- k : spring stiffness
- c : coefficient of the damper
- $R(t)$: external force (input)
- ℓ_u : unstretched length of spring
- x_{eq} : equilibrium position of mass (hanging position)
- x : position of mass *relative to* the equilibrium position (output)
- \dot{x} : velocity of mass
- \ddot{x} : acceleration of mass



Static Equilibrium

Using the equations of *static equilibrium*, it can be shown that the *equilibrium position* of the system under its own weight is

$$\boxed{x_{eq} = mg / k} . \quad (1)$$

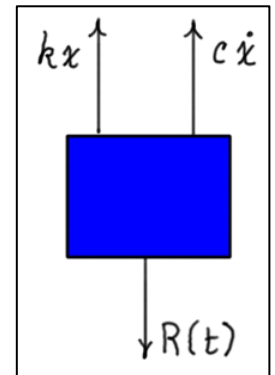


Equation of Motion

Using Newton's second law, it can be shown that the *differential equation of motion* is

$$\boxed{m \ddot{x} + c \dot{x} + k x = R(t)} . \quad (2)$$

Here, x is measured from the *equilibrium position*. Note that *static forces are not present* in this equation. The solution of this equation describes the *forced response* of the system. The *free response* of the system is described by solving the equation with $R(t) \equiv 0$.



System Parameters

The system mass m , spring stiffness k , and damping coefficient c are the *system parameters*. The first two are usually easier to measure (or estimate) than the third.

System Characteristic Equation and Response

The *differential equation* of *motion* represents a *mathematical model* of the system. It has *three types* of *solutions* depending on the values of the parameters m , c , and k . The type (or *character*) of solution is determined by the roots of the *characteristic equation* of the system. For the spring-mass-damper system, it can be shown that the characteristic equation is

$$\boxed{s^2 + (c/m)s + (k/m) = 0} \quad \text{or} \quad \boxed{s^2 + (2\zeta\omega_n)s + \omega_n^2 = 0}$$

where

$$\boxed{\omega_n = \sqrt{k/m}} \text{ is the } \textit{natural frequency} \text{ of the system}$$

$$\boxed{\zeta = c / 2\sqrt{mk}} \text{ is the } \textit{damping ratio}$$

In general, the *roots* of the *characteristic equation* can be written in the form

$$\boxed{s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}.$$

These roots may be *real* or *complex* depending on the value of ζ . The following table shows the *three types* of *motion* that are possible.

Case	Type of Roots	Type of Motion	Form of Solution
$\zeta < 1$	Complex conjugates	<i>Under-damped</i>	$x(t) = Ae^{-\zeta\omega_n t} (\cos(\omega_d t + \phi))$
$\zeta > 1$	Real, unequal	<i>Over-damped</i>	$x(t) = Ae^{s_1 t} + Be^{s_2 t}$
$\zeta = 1$	Real, equal	<i>Critically damped</i>	$x(t) = Ae^{-\zeta\omega_n t} + B t e^{-\zeta\omega_n t}$