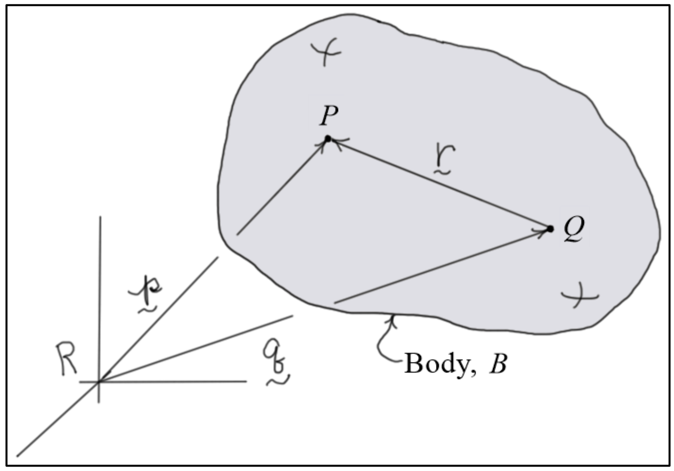
Intermediate Dynamics
Derivatives of a Vector in Two Different Reference Frames – "The Derivative Rule"

Motivation

It is often convenient to express vectors in terms of local (or rotating) unit vector sets (reference frames). For example, consider the position vector $\underset{\sim}{r}_{P/Q}$ the position vector of P relative to Q shown in the diagram. This vector describes the relative position of P and Q, two points ***fixed*** in the rigid body B. As such, it is most easily described in terms of the unit vector set $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ fixed in (and rotates with) body B.



To describe the ***relative motion*** of P and Q, the position vector $\underset{\sim}{r}_{P/Q}$ must be differentiated in the fixed reference frame (unit vector set $R:(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k})$). This can be done in one of two ways:

1) express $\underset{\sim}{r}_{P/Q}$ in terms of $R:(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k})$, and then ***differentiate***, or

2) express $\underset{\sim}{r}_{P/Q}$ in terms of $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$, and then ***use*** the ***derivative rule***.

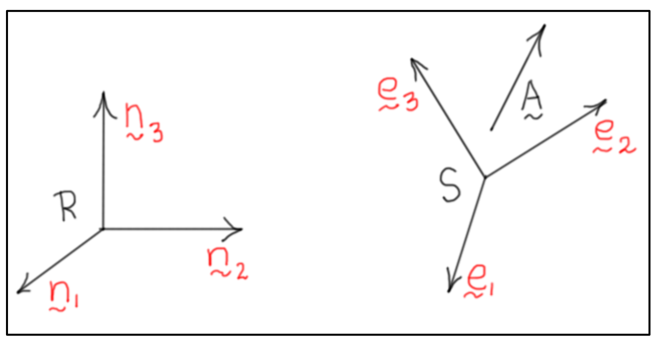
The Derivative Rule

Given the two reference frames

$R:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3)$ (rotating frame)

$S:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ (rotating frame),

the derivatives of ***any*** vector $\underset{\sim}{A}$ in the two reference frames are related by the following rule



$$\boxed{\frac{{}^{R}d\underset{\sim}{A}}{dt}=\frac{{}^{S}d\underset{\sim}{A}}{dt}+({}^{R}\underset{\sim}{\omega}_S\times\underset{\sim}{A})}$$

Here, ${}^{R}\underset{\sim}{\omega}_S$ is the angular velocity of frame S relative to the frame R.

Derivation

Consider the vector $\underset{\sim}{A}$ and the two reference frames R and S as shown. Suppose, for convenience, $\underset{\sim}{A}$ is expressed in terms of the unit vectors of frame S. That is,

$$\boxed{\underset{\sim}{A} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3}$$

Then, the derivative of $\underset{\sim}{A}$ in the reference frame R can be computed as follows

$$\begin{aligned}
\frac{{}^{R}d\underset{\sim}{A}}{dt} &= \underbrace{(\dot{a}_1 \underset{\sim}{e}_1 + \dot{a}_2 \underset{\sim}{e}_2 + \dot{a}_3 \underset{\sim}{e}_3)} + a_1 \frac{{}^{R}d\,\underset{\sim}{e}_1}{dt} + a_2 \frac{{}^{R}d\,\underset{\sim}{e}_2}{dt} + a_3 \frac{{}^{R}d\,\underset{\sim}{e}_3}{dt} \\
&= \frac{{}^{S}d\underset{\sim}{A}}{dt} + a_1({}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{e}_1) + a_2({}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{e}_2) + a_3({}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{e}_3) \\
&= \frac{{}^{S}d\underset{\sim}{A}}{dt} + {}^{R}\underset{\sim}{\omega}_S \times (a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3) \\
&= \frac{{}^{S}d\underset{\sim}{A}}{dt} + ({}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{A})
\end{aligned}$$

Here ${}^{R}\underset{\sim}{\omega}_S$ is the angular velocity of the reference frame $S:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ relative to the reference frame $R:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$.