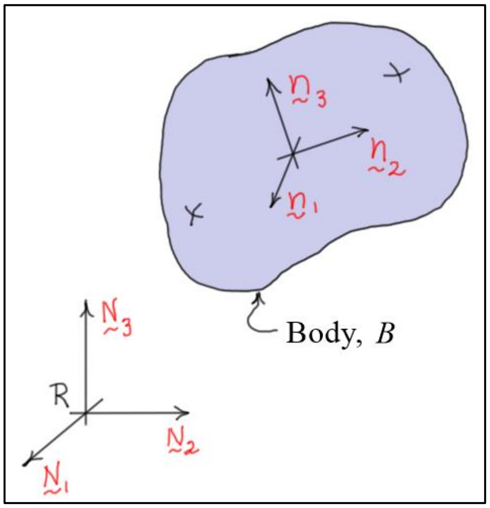
Multibody Dynamics
Orientation Angles of a Rigid Body in Three Dimensions

To describe the ***general orientation*** of a rigid body in three dimensions, consider the rigid body shown in the figure at the right. Here there are two reference frames – the base frame $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$, and the body-fixed frame $B:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$. In any arbitrary position none of the unit vectors of the two frames are aligned. There are generally two methods for describing the orientation of *B* relative to the base frame *R*.



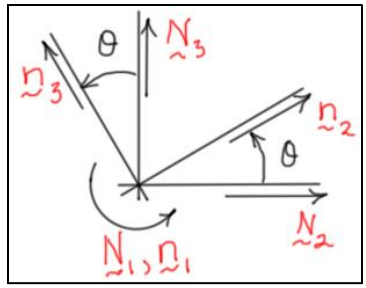
The first (and most commonly used) method of orienting a body in three dimensions involves the use of ***orientation angles***. These are easy to visualize, but they are ***not unique***, and they give rise to ***singularities*** in certain positions. The second method involves the use of ***Euler*** (or Euler-like) ***parameters***. These are not easy to visualize; however, they are unique, and they have ***no singularities***. The following notes discuss the use of orientation angles to describe angular position and motion of rigid bodies.

Simple Rotations

Simple rotations are defined as ***right-handed*** (or dextral) rotations about a single axis. For example, assume initially that the directions $(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ are aligned with the directions $(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$. Then, an *X*-rotation is defined as a ***right-handed*** (or dextral) rotation of *B* about $\underset{\sim}{N}_1$ (or $\underset{\sim}{n}_1$), a *Y*-rotation as a ***right-handed*** rotation about $\underset{\sim}{N}_2$ (or $\underset{\sim}{n}_2$), and a *Z*-rotation as a ***right-handed*** rotation about $\underset{\sim}{N}_3$ (or $\underset{\sim}{n}_3$). For each of these simple rotations, the unit vectors of the two reference frames can be related to each other using matrix equations.

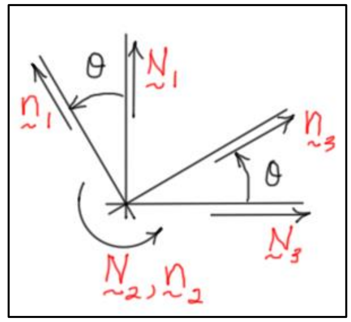
$\underline{X\text{-Rotation:}}$

$$\begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix}$$



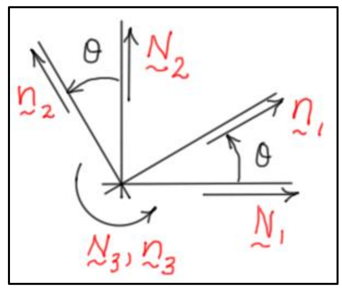
$\underline{Y\text{-Rotation:}}$

$$\begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix}$$



$\underline{Z\text{-Rotation:}}$

$$\begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix}$$

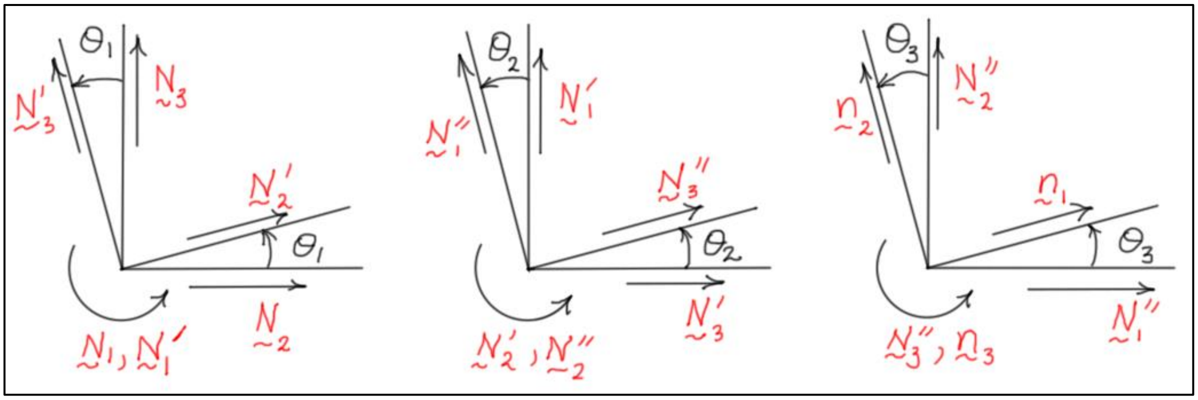


Here, S_θ and C_θ represent the ***sine*** and ***cosine*** of the angle of rotation θ.

The coefficient matrices in the above equations are called “***transformation***” or “***rotation***” matrices. They are ***orthogonal*** matrices with a ***determinant*** of +1. As with all orthogonal matrices, the ***inverses*** of these matrices are simply their ***transposes***. Hence, it is easy to invert the above equations.

General Orientations

A rigid body can be moved into any arbitrary orientation (relative to a base frame) using a sequence of three simple rotations. These rotations can occur about the base-frame axes or the body-frame axes. One common example is a ***body-fixed*** 1-2-3 rotation sequence. (Here, "1-2-3" has been used to stand for $\underset{\sim}{n}_1$, $\underset{\sim}{n}_2$, $\underset{\sim}{n}_3$ rotations.) To work through the sequence of rotations, ***intermediate reference frames*** may be introduced as shown below.



The ***matrix equations*** for the three rotations can be written as

$$\begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_1 & S_1 \\ 0 & -S_1 & C_1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = [R_1] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} \qquad \begin{Bmatrix} \underset{\sim}{N}_1'' \\ \underset{\sim}{N}_2'' \\ \underset{\sim}{N}_3'' \end{Bmatrix} = \begin{bmatrix} C_2 & 0 & -S_2 \\ 0 & 1 & 0 \\ S_2 & 0 & C_2 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix} = [R_2] \begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix}$$

$$\begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1'' \\ \underset{\sim}{N}_2'' \\ \underset{\sim}{N}_3'' \end{Bmatrix} = [R_3] \begin{Bmatrix} \underset{\sim}{N}_1'' \\ \underset{\sim}{N}_2'' \\ \underset{\sim}{N}_3'' \end{Bmatrix}$$

As before, S_i and C_i represent the ***sine*** and ***cosine*** of the orientation angle θ_i.

These equations can be ***combined*** to form a single matrix relationship between the base-fixed and the body-fixed unit vectors as follows

$$\boxed{\begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = [R_3][R_2][R_1] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = [R] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}}$$

So, for a body-fixed 1-2-3 rotation sequence, the ***transformation matrix*** that relates the unit vectors in the body reference frame to those in the base reference frame is

$$\boxed{[R] = [R_3][R_2][R_1] = \begin{bmatrix} C_2C_3 & C_1S_3 + S_1S_2C_3 & S_1S_3 - C_1S_2C_3 \\ -C_2S_3 & C_1C_3 - S_1S_2S_3 & S_1C_3 + C_1S_2S_3 \\ S_2 & -S_1C_2 & C_1C_2 \end{bmatrix}}$$

Here, the matrices $[R_i]$ are defined in the above equations. Like the individual rotation matrices $[R_i]$, the matrix $[R]$ is an ***orthogonal matrix*** whose ***determinant*** is +1. So, again it is easy to ***invert*** the relationship between the unit vector sets.

Note: Transformation matrices for many different combinations of rotations are given in Appendix I of the text *Spacecraft Dynamics* by T. R. Kane, P. W. Likins, and D. A. Levinson, McGraw-Hill, 1983.

<u>Relationship Between Vector Components in the Base and Body Frames</u>

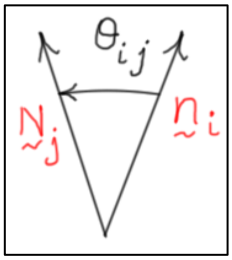
Given a vector $\underset{\sim}{A}$ expressed in two different reference frames

$$\boxed{\underset{\sim}{A} = A_1 \underset{\sim}{N}_1 + A_2 \underset{\sim}{N}_2 + A_3 \underset{\sim}{N}_3 = a_1 \underset{\sim}{n}_1 + a_2 \underset{\sim}{n}_2 + a_3 \underset{\sim}{n}_3},$$

the vector components can be related using the transformation matrix as follows

$$\boxed{\{A\} = [R]^T \{a\}} \qquad \text{or} \qquad \boxed{\{a\} = [R]\{A\}}$$

<u>Transformation Matrices and Direction Cosines</u>



The elements of a transformation matrix that relates the unit vectors of two different reference frames are the ***direction cosines*** of the various unit vector pairs. Given that R_{ij} represents the elements of the transformation matrix $[R]$, then

$$\boxed{R_{ij} = \underset{\sim}{n}_i \cdot \underset{\sim}{N}_j = C_{\theta_{ij}}}$$

Here, $C_{\theta_{ij}}$ represents the ***cosine*** of the angle between the unit vectors $\underset{\sim}{n}_i$ and $\underset{\sim}{N}_j$.