Elementary Dynamics
Curvilinear Motion – Rectangular Components

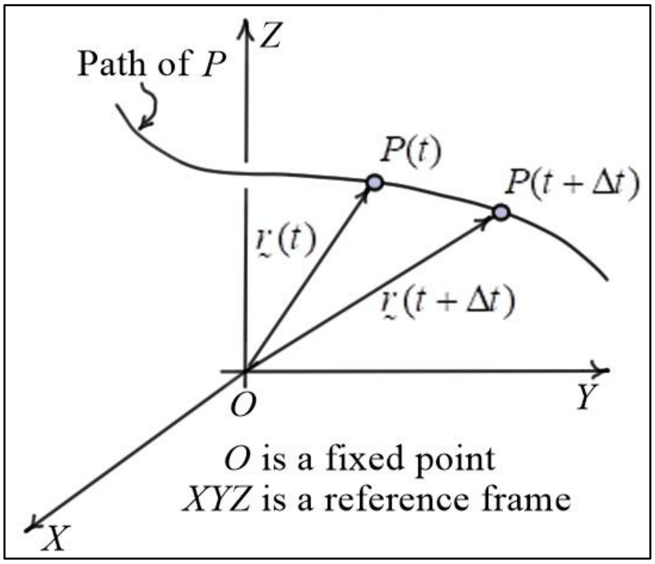
General Concepts:

Position, Velocity, and Acceleration

If a particle does not move in a straight line, then its motion is said to be ***curvilinear***. Given $\underset{\sim}{r}(t)$ the position vector of a particle P, the velocity and acceleration of P are defined to be

$$\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} \qquad \text{and} \qquad \underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt}.$$

The ***velocity*** $\underset{\sim}{v}$ is ***always tangent to the path*** of P. The ***acceleration*** $\underset{\sim}{a}$ is generally not tangent to the path.



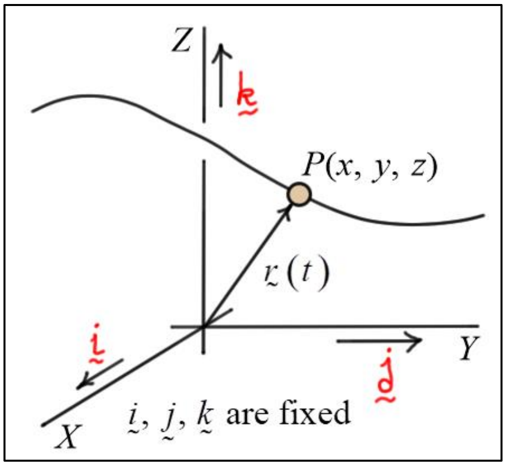
Rectangular Components

If we use rectangular components, then the position, velocity, and acceleration vectors may be written as

$$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$$

$$\underset{\sim}{v}(t) = \dot{x}(t)\underset{\sim}{i} + \dot{y}(t)\underset{\sim}{j} + \dot{z}(t)\underset{\sim}{k}$$

$$\underset{\sim}{a}(t) = \ddot{x}(t)\underset{\sim}{i} + \ddot{y}(t)\underset{\sim}{j} + \ddot{z}(t)\underset{\sim}{k}$$



Note that the methods for straight-line (rectilinear) motion can be applied in each direction.

Example: The Projectile Problem

If we ***neglect air resistance***, the motion of a projectile can be analyzed using the equations for constant acceleration. The ***horizontal motion*** (X-direction) occurs at a ***constant velocity***, and the ***vertical motion*** (Y-direction) occurs at a ***constant acceleration***. The equations that apply in the X and Y directions are

X-direction: (constant velocity, v_{x_0})

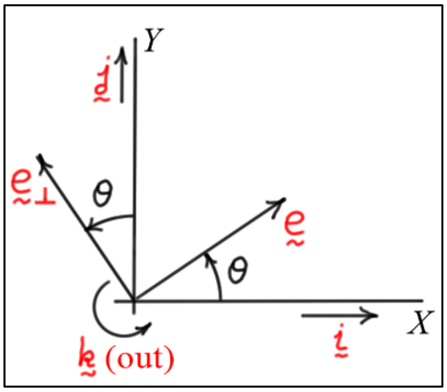
$$x(t) = x_0 + v_{x_0}\, t$$

Y-direction: (constant acceleration, $-g$)

$$v_y(t) = v_{y_0} - gt \qquad y(t) = y_0 + v_{y_0} t - \tfrac{1}{2} g t^2 \qquad v_y^2 = v_{y_0}^2 - 2g(y - y_0)$$

Derivative of a Rotating Unit Vector in Two Dimensions

In later notes on curvilinear motion, ***rotating unit vectors*** will be used instead of ***fixed unit vectors***. To this end, consider the rotating unit vector $\underset{\sim}{e}$ shown in the diagram. It can be written in terms of the fixed unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as



$$\underset{\sim}{e} = \cos(\theta)\,\underset{\sim}{i} + \sin(\theta)\,\underset{\sim}{j}$$

Here, angle θ varies with time as $\underset{\sim}{e}$ rotates. ***Differentiating*** with respect to time using the chain rule gives

$$\dot{\underset{\sim}{e}} \triangleq \frac{d\,\underset{\sim}{e}}{dt} = -\dot{\theta}\sin(\theta)\,\underset{\sim}{i} + \dot{\theta}\cos(\theta)\,\underset{\sim}{j} = \dot{\theta}\left(-\sin(\theta)\,\underset{\sim}{i} + \cos(\theta)\,\underset{\sim}{j}\right) \qquad \Rightarrow \boxed{\dot{\underset{\sim}{e}} = \dot{\theta}\,\underset{\sim}{e}_{\perp}}$$

The unit vector $\underset{\sim}{e}_{\perp}$ is rotated 90 degrees counterclockwise from $\underset{\sim}{e}$. Using the vector cross product, we note that $\underset{\sim}{e}_{\perp}$ can be written as $\underset{\sim}{e}_{\perp} = \underset{\sim}{k} \times \underset{\sim}{e}$. Using this observation, write

$$\dot{\underset{\sim}{e}} = \dot{\theta}\left(\underset{\sim}{k} \times \underset{\sim}{e}\right) = \left(\dot{\theta}\,\underset{\sim}{k}\right) \times \underset{\sim}{e} \qquad \Rightarrow \boxed{\dot{\underset{\sim}{e}} = \underset{\sim}{\omega} \times \underset{\sim}{e}}$$

Here, $\underset{\sim}{\omega} \triangleq \dot{\theta}\,\underset{\sim}{k}$ is the ***angular velocity*** of the rotating unit vector set $\left(\underset{\sim}{e}, \underset{\sim}{e}_{\perp}\right)$.