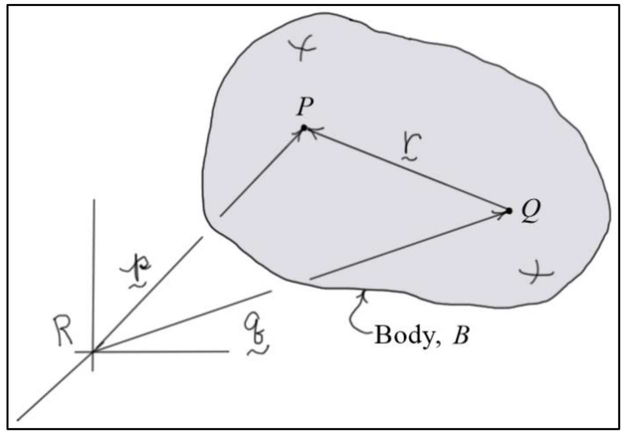
Intermediate Dynamics
Relative Kinematics of Two Points Fixed on a Rigid Body

General Concept

Consider the three-dimensional motion of a rigid body B as shown in the diagram. The points P and Q represent two points that are fixed in the body. The velocities and accelerations of P and Q in the reference frame R are related as follows



$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_Q + {}^R\underset{\sim}{v}_{P/Q} = {}^R\underset{\sim}{v}_Q + ({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r})}$$

and

$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_Q + {}^R\underset{\sim}{a}_{P/Q} = {}^R\underset{\sim}{a}_Q + \left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + {}^R\underset{\sim}{\omega}_B \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)}$$

These equations are easily verified using "the derivative rule" discussed in previous notes.

Derivation

1. The position vector of point P relative to the reference frame R can be written as $\underset{\sim}{p} = \underset{\sim}{q} + \underset{\sim}{r}$.

   Differentiating this equation and using "the derivative rule" gives

$$\begin{aligned}
{}^R\underset{\sim}{v}_P &= \frac{{}^R d\underset{\sim}{p}}{dt} = \frac{{}^R d\underset{\sim}{q}}{dt} + \frac{{}^R d\underset{\sim}{r}}{dt} \\
&= {}^R\underset{\sim}{v}_Q + \frac{{}^B d\underset{\sim}{r}}{dt} + {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r} \qquad \Rightarrow \qquad \boxed{\underset{\sim}{v}_P = \underset{\sim}{v}_Q + \underset{\sim}{v}_{P/Q} = \underset{\sim}{v}_Q + (\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q})} \\
&= {}^R\underset{\sim}{v}_Q + {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}
\end{aligned}$$

Here,

$$\boxed{\frac{{}^R d\underset{\sim}{r}}{dt} = {}^R\underset{\sim}{v}_{P/Q} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}} \qquad \text{(velocity of } P \text{ relative to } Q \text{ in } R,\ {}^R\underset{\sim}{v}_{P/Q})$$

2. Differentiating the velocity equation and again using "the derivative rule" gives

$$\begin{aligned}
{}^R\underset{\sim}{a}_P &= \frac{{}^R d}{dt}({}^R\underset{\sim}{v}_P) = \frac{{}^R d}{dt}({}^R\underset{\sim}{v}_Q) + \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right) \\
&= {}^R\underset{\sim}{a}_Q + \left(\frac{{}^R d}{dt}({}^R\underset{\sim}{\omega}_B) \times \underset{\sim}{r}\right) + \left({}^R\underset{\sim}{\omega}_B \times \frac{{}^R d\underset{\sim}{r}}{dt}\right) \\
&= {}^R\underset{\sim}{a}_Q + \left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + {}^R\underset{\sim}{\omega}_B \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)
\end{aligned}$$

or

$$\boxed{{}^{R}\underset{\sim}{a}_{P} = {}^{R}\underset{\sim}{a}_{Q} + {}^{R}\underset{\sim}{a}_{P/Q} = {}^{R}\underset{\sim}{a}_{Q} + \left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)}$$

Here ${}^{R}\underset{\sim}{a}_{P/Q}$ is the acceleration of P with respect to Q in R, and by inspection of the above equation, it is defined to be

$$\boxed{{}^{R}\underset{\sim}{a}_{P/Q} = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{P/Q}\right) = \left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)}$$

<u>Notes</u>

1. The above formulae may be applied ***recursively*** to calculate motions of ***remote points*** within a mechanical system.
2. Calculation of velocities and accelerations ***does not require differentiation***, only multiplication and addition.