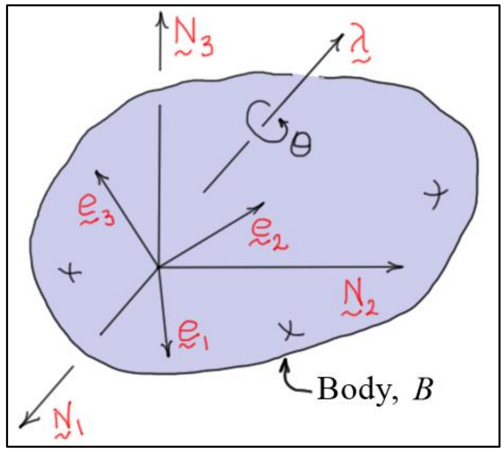
Multibody Dynamics
Orientation of a Rigid Body Using Euler Parameters
(Reference: T.R. Kane, P.W. Likins, D.A. Levinson, *Spacecraft Dynamics*, McGraw-Hill, 1983.

Euler's Theorem on Rotation

Consider the rigid body shown in the figure at the right. Let $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$ represent the ***base reference frame*** and $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ represent the ***body-fixed reference frame*** and assume initially that the two frames are ***aligned***. Then, ***Euler's Theorem on Rotation*** states that the rigid body $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ can be moved into any ***arbitrary orientation*** relative to the base frame by a rotation about a single axis. Here, θ represents the angle of rotation, and the ***unit vector*** $\underset{\sim}{\lambda}$ represents the direction (or axis) of rotation.



Euler Parameters

The unit vector $\underset{\sim}{\lambda}$ and the angle θ can be related to a set of ***four parameters*** called the Euler ***parameters***. First, let $\underset{\sim}{\lambda}$ be expressed in terms of the base-frame unit vectors as

$$\boxed{\underset{\sim}{\lambda} = \lambda_1 \underset{\sim}{N}_1 + \lambda_2 \underset{\sim}{N}_2 + \lambda_3 \underset{\sim}{N}_3}$$

Then, the four Euler parameters may be defined as

$$\boxed{\varepsilon_i = \lambda_i \sin(\theta/2)} \qquad (i=1,2,3) \qquad \text{and} \qquad \boxed{\varepsilon_4 = \cos(\theta/2)}$$

Notes

1. The Euler parameters are ***not independent***, because $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$.
2. It can be shown that

$$\begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} = [R] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = \begin{bmatrix} (\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) & 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) & (-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) & (-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$$

3. ***No singularities*** exist in these kinematic equations, so many computer programs use Euler parameters to avoid computational singularities; however, they may communicate with the "user" using orientation angles which are easier to ***visualize***.