

Introductory Control Systems

Linearization of System Models

The *differential* or *algebraic* equation that models the behavior of a system may be *linear* or *nonlinear*. If it is nonlinear, it may be possible to *linearize* the equation about some *nominal operating condition* (or *set point*). The focus here is on linearization of *algebraic models only*.

Linearization of Functions of a Single Variable

If the output of a system (y) is related to the input of the system (x) with a nonlinear algebraic function $y = f(x)$, an approximate linear model of the system behavior can be developed using a Taylor series. First, identify a set point (nominal condition), and then expand $f(x)$ in a *Taylor series* around that point. Referring to the set point as an *equilibrium position* x_{eq} , write

$$\begin{aligned} f(x_{eq} + \Delta x) &= f(x_{eq}) + \Delta x \left[\frac{df}{dx} \right]_{x=x_{eq}} + \frac{(\Delta x)^2}{2} \left[\frac{d^2 f}{dx^2} \right]_{x=x_{eq}} + \dots \\ &\approx f(x_{eq}) + \Delta x \left[\frac{df}{dx} \right]_{x=x_{eq}} \end{aligned}$$

Here Δx represents an *excursion* or *deviation* from the equilibrium position (set point). If the *excursions* are *small*, then the approximation as stated in the second equation can be used. In this latter case, write

$$\Delta f(x) = f(x_{eq} + \Delta x) - f(x_{eq}) = m \Delta x$$

where

$$m = \left. \frac{df}{dx} \right|_{x=x_{eq}}$$

This is a *linear* relationship between *changes* in f and *changes* in x .

Note: It is up to the analyst to determine the *range* of Δx over which the linear model is *acceptably accurate*. This range could be *quite large* for *mildly nonlinear* functions, but it could be *quite small* for *strongly nonlinear* functions.

Linearization of Functions of Many Variables

Given a nonlinear function $y = f(x_1, x_2, \dots, x_n) = f(\underline{x})$, expand $f(\underline{x})$ in a *Taylor series* around the *equilibrium position*, say $\underline{x}_{eq} = ((x_1)_{eq}, (x_2)_{eq}, \dots, (x_n)_{eq})$

$$\begin{aligned} f(\underline{x}_{eq} + \Delta \underline{x}) &= f(\underline{x}_{eq}) + \sum_{i=1}^n \Delta x_i \left[\frac{\partial f}{\partial x_i} \right]_{\underline{x}=\underline{x}_{eq}} + \dots \\ &\approx f(\underline{x}_{eq}) + \sum_{i=1}^n \Delta x_i \left[\frac{\partial f}{\partial x_i} \right]_{\underline{x}=\underline{x}_{eq}} \end{aligned}$$

Here the vector $\Delta \underline{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)$ represents an *excursion* from the *equilibrium position*.

As before, if the excursions are *small*, then the approximation as stated in the second equation can be used. In this latter case, write

$$\Delta f(\underline{x}) = f(\underline{x}_{eq} + \Delta \underline{x}) - f(\underline{x}_{eq}) = \sum_{i=1}^n m_i \Delta x_i$$

where

$$m_i = \left. \frac{\partial f}{\partial x_i} \right|_{\underline{x}=\underline{x}_{eq}}$$

This is a *linear* relationship between *changes* in f and *changes* in the elements of vector \underline{x} . As before, it is up to the analyst to determine acceptable ranges of the Δx_i ($i = 1, \dots, n$).