**Multibody Dynamics**
**Time Derivative of the (Coordinate) Transformation Matrices**

## Matrix Form of the Derivative of a Vector Fixed in a Rigid Body

Consider a body $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ moving in a fixed reference frame $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$. If $\underset{\sim}{r}$ is a vector fixed in the body *B*, then the derivative of $\underset{\sim}{r}$ may be written as

$$\boxed{\frac{{}^R d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}} \tag{1}$$

When performing the cross product, the individual vectors and the resulting cross product can be expressed in any reference frame. In the following paragraphs, "primes" indicate vector components in body $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$, and "no primes" indicate vector components in $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$. Also, let $[R]$ be the transformation matrix that relates the two sets of unit vectors as defined by the equation

$$\boxed{\{\underset{\sim}{e}\} = [R]\{\underset{\sim}{N}\}} \tag{2}$$

**<u>Case 1</u>**: $\dot{\underset{\sim}{r}}$ expressed in $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$, but ${}^R\underset{\sim}{\omega}_B$ and $\underset{\sim}{r}$ expressed in $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$

In this case, write $\boxed{\dot{\underset{\sim}{r}} = \sum_{i=1}^{3} \dot{r}_i\, \underset{\sim}{N}_i}$, $\boxed{{}^R\underset{\sim}{\omega}_B = \sum_{i=1}^{3} \omega'_i\, \underset{\sim}{e}_i}$, and $\boxed{\underset{\sim}{r} = \sum_{i=1}^{3} r'_i\, \underset{\sim}{e}_i}$.

The three sets of vector components are related by the matrix form of Eq. (1).

$$\dot{\underset{\sim}{r}} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r} \qquad \rightarrow \qquad \boxed{\{\dot{r}\} = [R]^T\left([\tilde{\omega}']\{r'\}\right) = \left([R]^T[\tilde{\omega}']\right)\{r'\}} \tag{3}$$

**<u>Case 2</u>**: $\dot{\underset{\sim}{r}}$ and ${}^R\underset{\sim}{\omega}_B$ expressed in $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$, but $\underset{\sim}{r}$ expressed in $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$

In this case, write $\boxed{\dot{\underset{\sim}{r}} = \sum_{i=1}^{3} \dot{r}_i\, \underset{\sim}{N}_i}$, $\boxed{{}^R\underset{\sim}{\omega}_B = \sum_{i=1}^{3} \omega_i\, \underset{\sim}{N}_i}$, and $\boxed{\underset{\sim}{r} = \sum_{i=1}^{3} r'_i\, \underset{\sim}{e}_i}$.

The three sets of vector components are again related by the matrix form of Eq. (1)

$$\dot{\underset{\sim}{r}} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r} \qquad \rightarrow \qquad \boxed{\{\dot{r}\} = [\tilde{\omega}]\left([R]^T\{r'\}\right) = \left([\tilde{\omega}][R]^T\right)\{r'\}} \tag{4}$$

## Time Derivative of the Transformation Matrices

The above results can be used to determine ***two different forms*** of the ***time derivative*** of the transformation matrix $[R]$. To do this, first note that the components of position vector $\underset{\sim}{r}$ in the two different reference frames are related as follows

$$\{r\} = [R]^T \{r'\}$$

This matrix equation can be ***differentiated*** directly to give

$$\boxed{\{\dot{r}\} = \left[\dot{R}\right]^T \{r'\} + [R]^T \{\dot{r}'\} = \left[\dot{R}\right]^T \{r'\}}$$

Here, advantage is taken of the fact that since $\underset{\sim}{r}$ is fixed in the body, $\boxed{\{\dot{r}'\} = \{0\}}$. Comparing this result with Eqs. (3) and (4) gives the two forms of $[\dot{R}]$.

$$\boxed{\left[\dot{R}_K\right]^T = [R_K]^T \left[\tilde{\omega}'_K\right]} \quad \text{and} \quad \boxed{\left[\dot{R}_K\right]^T = \left[\tilde{\omega}_K\right][R_K]^T}$$

or

$$\boxed{\left[\dot{R}_K\right] = \left[\tilde{\omega}'_K\right]^T [R_K]} \quad \text{and} \quad \boxed{\left[\dot{R}_K\right] = [R_K]\left[\tilde{\omega}_K\right]^T}$$