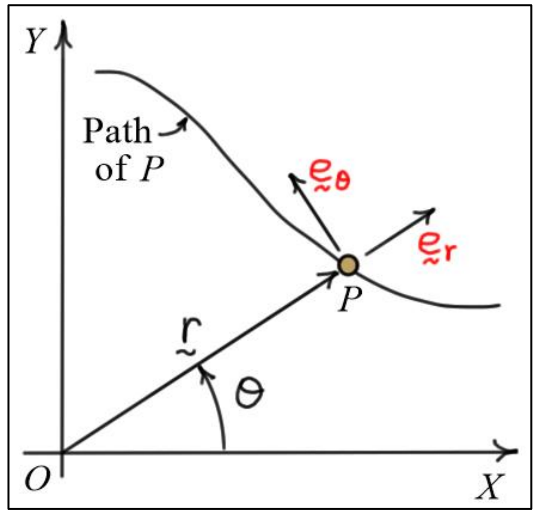
Elementary Dynamics
Curvilinear Motion – Radial and Transverse Components

Radial and Transverse Components

Another way to describe the motion of *P* as it moves along a curved path is to use ***radial*** and ***transverse*** components. Here, we define the unit vector $\underset{\sim}{e}_r$ to point radially outward from *O* to *P*. The ***position vector*** of *P* can then be written as

$$\boxed{\underset{\sim}{r} = r\,\underset{\sim}{e}_r}$$



To find an expression for the ***velocity*** of *P*, we differentiate (using the product rule)

$$\underset{\sim}{v} = \dot{r}\underset{\sim}{e}_r + r\dot{\underset{\sim}{e}}_r = \dot{r}\underset{\sim}{e}_r + r\left(\dot{\theta}\underset{\sim}{k} \times \underset{\sim}{e}_r\right) = \dot{r}\,\underset{\sim}{e}_r + r\,\dot{\theta}\,\underset{\sim}{e}_\theta \quad \Rightarrow \quad \boxed{\underset{\sim}{v} = \dot{r}\,\underset{\sim}{e}_r + r\dot{\theta}\,\underset{\sim}{e}_\theta}$$

To find an expression for the ***acceleration*** of *P*, differentiate again using the product rule.

$$\underset{\sim}{a} = \ddot{r}\,\underset{\sim}{e}_r + \dot{r}\,\dot{\underset{\sim}{e}}_r + \dot{r}\dot{\theta}\underset{\sim}{e}_\theta + r\ddot{\theta}\underset{\sim}{e}_\theta + r\dot{\theta}\dot{\underset{\sim}{e}}_\theta$$

Here,

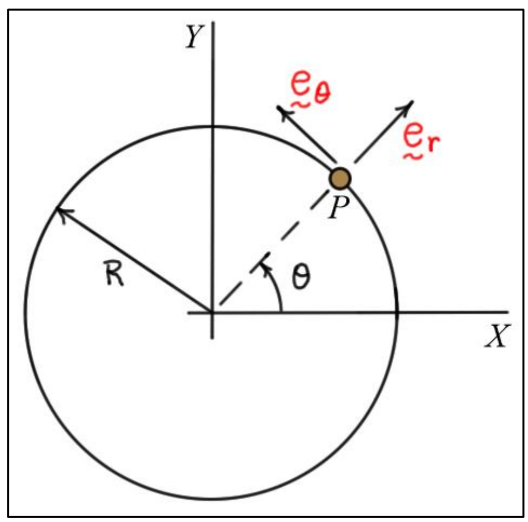
$$\dot{\underset{\sim}{e}}_r = \dot{\theta}\underset{\sim}{k} \times \underset{\sim}{e}_r = \dot{\theta}\underset{\sim}{e}_\theta \quad \text{and} \quad \dot{\underset{\sim}{e}}_\theta = \dot{\theta}\underset{\sim}{k} \times \underset{\sim}{e}_\theta = -\dot{\theta}\underset{\sim}{e}_r$$

Using these two results in the expression for the acceleration and collecting terms gives

$$\boxed{\underset{\sim}{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underset{\sim}{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underset{\sim}{e}_\theta}$$

Special Case: Circular Motion

During circular motion the distance between *O* and *P* remains ***constant*** and the transverse unit vector $\underset{\sim}{e}_\theta$ is ***tangent*** to the path.

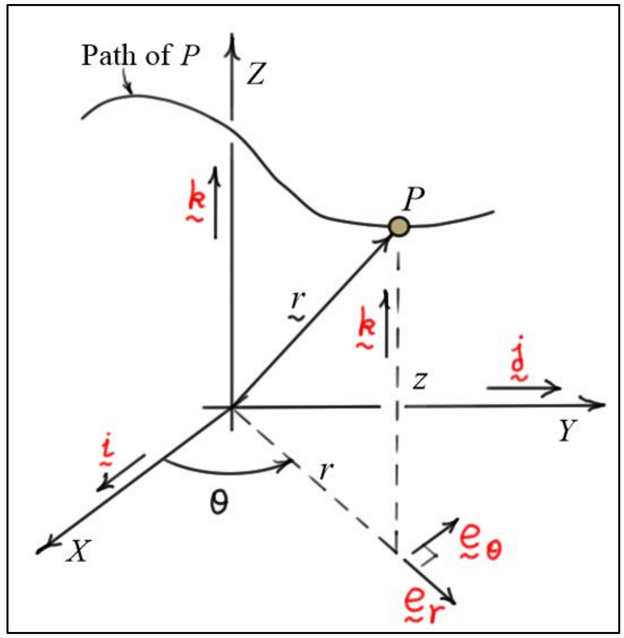


$$r = R = \text{constant} \qquad \Rightarrow \dot{r} = \ddot{r} = 0$$

$$\Rightarrow \boxed{\begin{array}{l} \underset{\sim}{v} = R\dot{\theta}\underset{\sim}{e}_\theta \\ \underset{\sim}{a} = \left(-R\dot{\theta}^2\right)\underset{\sim}{e}_r + \left(R\ddot{\theta}\right)\underset{\sim}{e}_\theta \end{array}}$$

Cylindrical Components

Cylindrical components represent the extension of the concept of ***radial*** and ***transverse*** components to three dimensions. The projection of P onto the XY plane is tracked with radial and transverse directions in that plane. Motion of P perpendicular to the XY plane is tracked by the Cartesian (rectangular) coordinate z. In this case, write



$$\underset{\sim}{r} = r\,\underset{\sim}{e}_r + z\,\underset{\sim}{k}$$

$$\underset{\sim}{v} = \dot{r}\,\underset{\sim}{e}_r + r\dot{\theta}\,\underset{\sim}{e}_\theta + \dot{z}\,\underset{\sim}{k}$$

$$\underset{\sim}{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underset{\sim}{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underset{\sim}{e}_\theta + \ddot{z}\,\underset{\sim}{k}$$