

Introductory Control Systems

Equations of Motion of a Simple Pendulum and Accelerating Simple Pendulum

Simple Pendulum

- A *simple pendulum* with end mass m is shown in the figure to the right.

The end mass is assumed to be connected to the support with a *light, slender, rigid rod* of length ℓ .

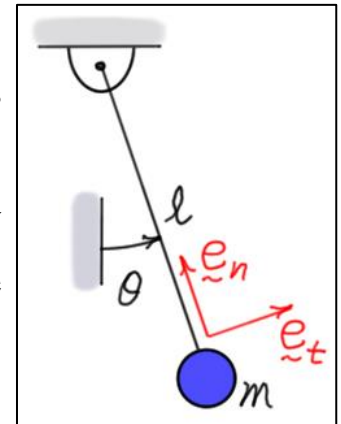
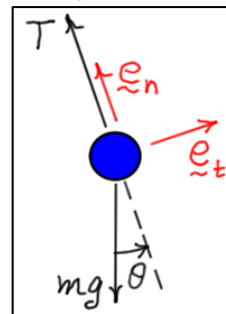
- The *differential equation of motion* of the pendulum can be found by summing forces in the tangential (\tilde{e}_t) direction. Referring to the free

body diagram, write

$$\sum F_t = ma_t \Rightarrow -mg \sin(\theta) = m\ell \ddot{\theta}$$

or

$$\ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0 \quad (1)$$



- The solution of this differential equation describes the movement of the pendulum for *any initial conditions*.

- Equilibrium Positions: These are found by setting $\ddot{\theta} = 0$ in the differential equation of motion (1). Hence, the pendulum has *two equilibrium positions*: $\theta_{eq} = 0, \pi$.

- Linearized Equation of Motion: $\theta_{eq} = 0$ To study *small motions* of the pendulum about the equilibrium position $\theta_{eq} = 0$, let $\theta = \theta_{eq} + \Delta\theta$ and linearize the equation of motion (1) about that position. This is done by linearizing the function $f(\theta) = \sin(\theta)$ about $\theta = 0$.

$$\Delta f = \left(\frac{df}{d\theta} \right)_{\theta=0} \Delta\theta = 1 \cdot \Delta\theta = \Delta\theta$$

- Hence, the *approximate linear equation of motion* for small motions about $\theta = 0$ is

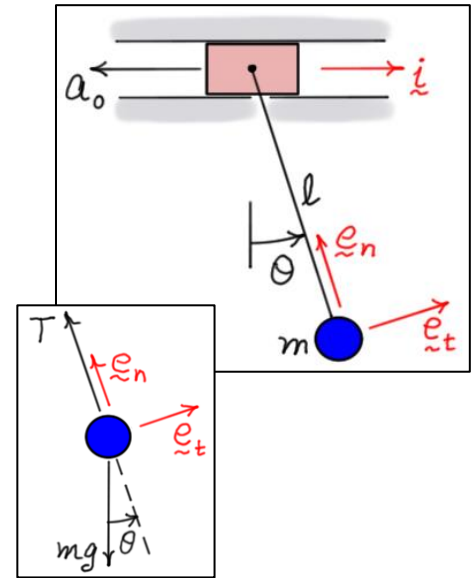
$$\Delta \ddot{\theta} + \left(\frac{g}{\ell} \right) \Delta\theta = 0$$

- The solution of this differential equation describes *small movements* of the pendulum about $\theta = 0$.

Accelerating Simple Pendulum

- An **accelerating simple pendulum** with end mass m is shown in the figure. The end mass is assumed to be connected to the accelerating base with a **light, slender, rigid rod** of length ℓ .
- The base **accelerates** at a **constant** rate a_0 to the **left**. The acceleration of mass m may be found using the **relative acceleration** equation as follows:

$$\begin{aligned} \underline{a}_m &= \underline{a}_A + \underline{a}_{m/A} = -a_0 \underline{i} + (\ell \ddot{\theta} \underline{e}_t + \ell \dot{\theta}^2 \underline{e}_n) \\ &= (\ell \ddot{\theta} - a_0 \cos(\theta)) \underline{e}_t + (\ell \dot{\theta}^2 + a_0 \sin(\theta)) \underline{e}_n \end{aligned}$$



- The **differential equation of motion** of the pendulum may be found by summing forces in the **tangential** (\underline{e}_t) direction. Referring to the **free body diagram**, write

$$\sum F_t = ma_t \Rightarrow -mg \sin(\theta) = m(\ell \ddot{\theta} - a_0 \cos(\theta)) \Rightarrow \ddot{\theta} + \left(\frac{g}{\ell} \sin(\theta) - \frac{a_0}{\ell} \cos(\theta) \right) = 0$$

- The solution of this differential equation describes the movement of the pendulum for **any initial conditions**.

- Equilibrium Positions:** As before, these positions are found by setting $\ddot{\theta} = 0$. Hence, the accelerating pendulum has two equilibrium positions: $\theta_{eq} = \tan^{-1}\left(\frac{a_0}{g}\right)$. For example, if

$$a_0 = 0.5g, \text{ then } \theta_{eq} = \tan^{-1}(0.5) = 26.57^\circ, 206.57^\circ.$$

- Linearized Equation of Motion:** $a_0 = 0.5g$ and $\theta_{eq} = 26.57^\circ$ To study **small motion** of the pendulum about $\theta_{eq} = 26.57^\circ$, let $\theta = \theta_{eq} + \Delta\theta$ and linearize the equation of motion about that position. This is done by linearizing the function $f(\theta) = \frac{g}{\ell} \left(\sin(\theta) - \frac{1}{2} \cos(\theta) \right)$ about $\theta_{eq} = 26.57^\circ$.

$$\Delta f(\theta) = \left(\frac{df}{d\theta} \Big|_{\theta=\theta_{eq}} \right) \Delta\theta = \left(\frac{g}{\ell} \left(\cos(\theta) + \frac{1}{2} \sin(\theta) \right) \right)_{\theta=\theta_{eq}} \Delta\theta = \left(1.118 \frac{g}{\ell} \right) \Delta\theta$$

- The *approximate linear equation of motion* for small motion about $\theta = 26.57^\circ$ is

$$\Delta\ddot{\theta} + \left(1.118\frac{g}{\ell}\right)\Delta\theta = 0$$

- The solution of this differential equation describes *small movements* of the pendulum about $\theta = 26.57^\circ$.
- Note the *multiplier* of $\frac{g}{\ell}$ is 1.0 for the non-accelerating pendulum and 1.118 for the accelerating pendulum. It can be shown that this means the *natural frequency* of the *accelerating pendulum* is *slightly higher* than the *non-accelerating pendulum*.