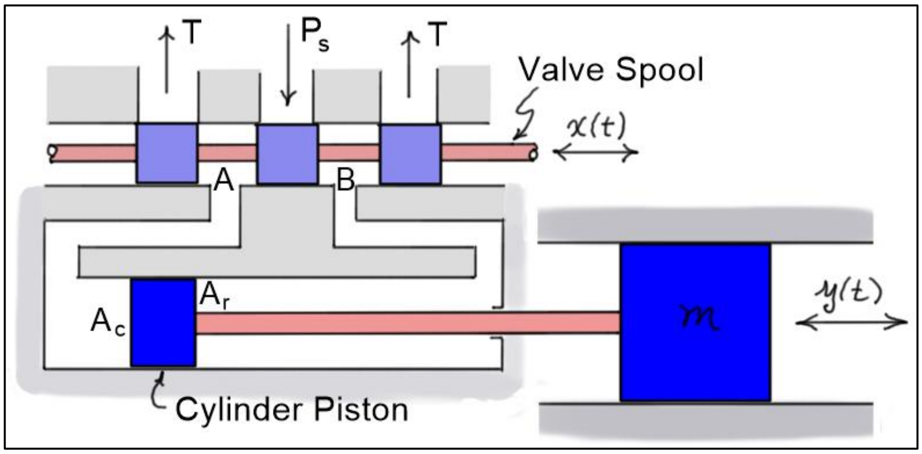
Introductory Motion and Control
Hydraulic Positioning System II

References: Dorf and Bishop, *Modern Control Systems*, 9th Ed., Prentice-Hall, 2001.
Parker *Design Engineer's Handbook: Vol. 1 Hydraulics*, Bulletin 0292-B1-H, 2001.

Positioning System

- Incompressible fluid
- A_c = cap end piston area
- A_r = rod end piston area
- m = mass of load
- b = damping coefficient
- P_s = constant supply pressure
- P = pressure on the piston
- p = ΔP, the change in P
- X = valve spool position
- x = ΔX, the change in X
- Y = load position
- y = ΔY, the change in Y



Operation

- If $X > 0$, then the ***pressure source*** is ***applied*** to the *A* ***port*** of the valve and the ***cap end*** of the cylinder causing the load to ***move*** to the ***right***. ***Return flow*** to the tank is though the *B* ***port***.
- If $X < 0$, then the ***pressure source*** is ***applied*** to the *B* ***port*** of the valve and the ***rod end*** of the cylinder causing the load to ***move*** to the ***left***. ***Return flow*** to the tank is though the *A* ***port***.

Flow Model

If $X > 0$, then the pressure source is applied to the A port of the valve. As a result, fluid flows into the cap end of the cylinder and out of the rod end. The flow rate through the valve is a function of X the spool position and the pressures on either side of the piston.

$$Q_A = g_A(X, P_A) \qquad \text{and} \qquad Q_B = g_B(X, P_B) \tag{1}$$

To simplify the model, Eqs. (1) can be ***linearized*** about some ***operational*** (set) ***points*** (X_0, P_{A0}) and (X_0, P_{B0}). This is done using a ***Taylor series expansion***. The change in flow rates can be written as

$$\boxed{\begin{aligned} q_A \triangleq \Delta Q_A &\approx \left(\frac{\partial g_A}{\partial X}\right)_{X_0, P_{A0}} \Delta X \; + \; \left(\frac{\partial g_A}{\partial P_A}\right)_{X_0, P_{A0}} \Delta P_A \\ &= \left(k_{xA}\right) x - \left(k_{pA}\right) p_A \end{aligned}} \tag{2}$$

$$\boxed{\begin{aligned} q_B \triangleq \Delta Q_B &\approx \left(\frac{\partial g_B}{\partial X}\right)_{X_0, P_{B0}} \Delta X \; + \; \left(\frac{\partial g_B}{\partial P_B}\right)_{X_0, P_{B0}} \Delta P_B \\ &= \left(k_{xB}\right) x + \left(k_{pB}\right) p_B \end{aligned}} \tag{3}$$

The coefficients k_{xA}, k_{xB}, k_{pA} and k_{pB} represent the ***derivatives*** of the function $g_A(X, P_A)$ and $g_B(X, P_B)$ with respect to X and P, respectively. The minus sign in the second of Eqs. (2), because the flow rate ***decreases*** as the pressure in the piston chamber ***increases***.

Assuming the fluid is ***incompressible***, the piston velocity can be related to the volumetric flow rates as follows.

$$\boxed{Q_A = A_c \dot{Y}} \qquad \text{and} \qquad \boxed{Q_B = A_r \dot{Y}} \tag{4}$$

Defining variations from the nominal conditions for each port ($Q = Q_0 + q$, $\dot{Y} = \dot{Y}_0 + \dot{y}$ and $Q_0 = A\dot{Y}_0$), then variations in the piston velocity can be related to changes in the volumetric flow rates as follows.

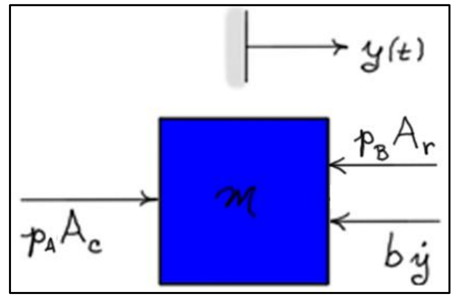
$$\boxed{q_A = A_c \dot{y}} \qquad \text{and} \qquad \boxed{q_B = A_r \dot{y}} \tag{5}$$

Combining Eqs. (2), (3) and (5) gives the following equations for the pressure changes at each port.

$$\boxed{p_A = (k_{xA} x - A_c \dot{y}) / k_{pA}} \qquad \text{and} \qquad \boxed{p_B = (-k_{xB} x + A_r \dot{y}) / k_{pB}} \tag{6}$$

Model for Piston Movement

Assuming $X > 0$ (flow is entering port *A* and leaving port *B*), ***Newton's second law*** can be used to write the equation of motion of the load.



$$\overset{+}{\rightarrow}\sum F = p_A A_c - p_B A_r - b\,\dot{y} = m\,\ddot{y} \qquad (7)$$

Rearranging the Eq. (7) and ***substituting*** for the ***pressure changes*** from Eqs. (6) gives

$$m\ddot{y} + \left(b + \frac{A_c^2}{k_{pA}} + \frac{A_r^2}{k_{pB}} \right)\dot{y} = \left(\frac{A_c k_{xA}}{k_{pA}} + \frac{A_r k_{xB}}{k_{pB}} \right) x \qquad (X > 0) \qquad (8)$$

If $X < 0$, then $p_A = (k_{xA}x - A_c\dot{y})/k_{pA}$, $p_B = (A_r\dot{y} - k_{xB}x)/k_{pB}$. Substituting these new pressure equations into Newton's law yields the ***same form of model*** equation as shown in Eq. (8). Note, however, ***the coefficients*** k_{xA}, k_{xB}, k_{pA} and k_{pB} ***will be different*** for the two cases, because the nominal pressures (about which the linearization is done) will be different.

Notes

- If we have a ***double rod cylinder***, then $A_c = A_r$, so the same model equation holds for motion in both directions. All coefficients will be the same.
- The motion described by Eq. (8) is ***second-order, over-damped motion***.
- If the mass of the load is small $(m \approx 0)$, then the motion is ***first-order***.

Orifice Flow

The volumetric flow rate through a ***sharp-edged orifice*** can be ***approximated*** using the equation

$$Q = C_d A\sqrt{2\Delta P/\rho} \qquad (9)$$

Here, A is the orifice area, ρ is the fluid mass density, ΔP is the pressure drop across the orifice, and C_d is a dimensionless discharge coefficient that depends on Reynolds number and the area reduction. Discharge coefficients in the range $0.6 \le C_d \le 0.8$ are reasonable for a large range of Reynolds numbers and area reductions.

Using Eq. (9) to ***model*** the ***flow into*** and ***out of*** the cylinder through the control valve, and assuming the ***area*** of the ***orifice*** is ***proportional*** to the ***valve spool displacement***, X, Eq. (9) can be written as

$$\boxed{Q_{\text{in}} = C_d \ell X \sqrt{2(P_s - P)/\rho}}$$

$$\boxed{Q_{\text{out}} = C_d \ell X \sqrt{2(P - P_T)/\rho} \approx C_d \ell X \sqrt{2P/\rho}}$$

where ℓ is a characteristic length such that $A = \ell X$. Using these two equations, the coefficients in Eqs. (2) and (3) can be estimated as shown below. ***For flow into one of the piston chambers***, the coefficients can be estimated to be

$$k_x = \left(\frac{\partial Q_{\text{in}}}{\partial X}\right)_{X_0, P_0} = \sqrt{\tfrac{2}{\rho}} C_d \ell \sqrt{P_s - P_0}$$

$$k_p = \left(\frac{\partial Q_{\text{in}}}{\partial P}\right)_{X_0, P_0} = \tfrac{1}{2} C_d \ell X_0 \sqrt{\tfrac{2}{\rho}} \left(P_s - P_0\right)^{-\frac{1}{2}} = \frac{C_d \ell X_0}{\sqrt{2\rho(P_s - P_0)}}$$

For flow out of one of the chambers, the coefficients can be estimated to be

$$k_x = \left(\frac{\partial Q_{\text{out}}}{\partial X}\right)_{X_0, P_0} = \sqrt{\tfrac{2}{\rho}} C_d \ell \sqrt{P_0}$$

$$k_p = \left(\frac{\partial Q_{\text{out}}}{\partial P}\right)_{X_0, P_0} = \tfrac{1}{2} C_d \ell X_0 \sqrt{\tfrac{2}{\rho}} \left(P_0\right)^{-\frac{1}{2}} = \frac{C_d \ell X_0}{\sqrt{2\rho P_0}}$$