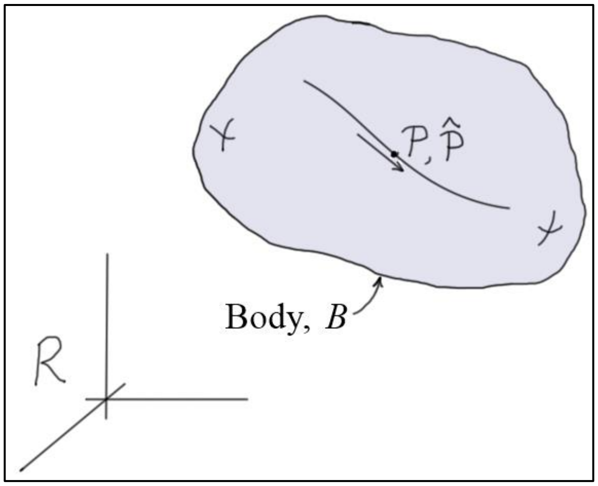
Intermediate Dynamics
Kinematics of a Point Moving on a Rigid Body

The kinematic analysis is now extended to include systems where interconnected bodies can ***rotate*** and ***translate*** relative to each other. In these cases, there is a need to describe the kinematics of points that are moving on (relative to) a rotating body. To analyze this motion, consider the figure shown at the right. Here,



R: a fixed reference frame

B: a moving rigid body

P: a point ***moving*** on B

$\hat{P}$: a point ***fixed*** on B that ***coincides*** with P at this instant of time

The ***velocity*** and ***acceleration*** of P can be written as

$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{\hat{P}} + {}^B\underset{\sim}{v}_P}$$

$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_{\hat{P}} + {}^B\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)}$$

Here, each of the terms are defined as follows.

${}^B\underset{\sim}{v}_P,\ {}^B\underset{\sim}{a}_P$: velocity and acceleration of P on B, assuming that B is fixed

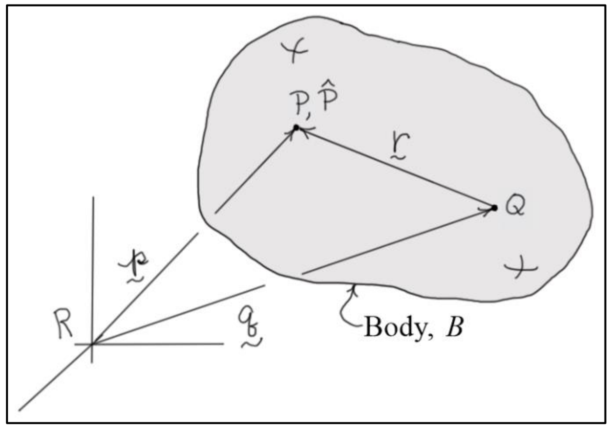
${}^R\underset{\sim}{v}_{\hat{P}},\ {}^R\underset{\sim}{a}_{\hat{P}}$: velocity and acceleration of $\hat{P}$ in R (recall that $\hat{P}$ is fixed on B)

$2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)$: Coriolis acceleration of P

Note: The ***velocity*** and ***acceleration*** of $\hat{P}$ can be determined using the formulae for points fixed on rigid bodies. See notes on "Relative Kinematics of Two Points Fixed on a Rigid Body".

Derivation

The results shown above can easily be shown by using "the derivative rule". Consider the rigid body shown in the diagram at the right. Here,



- R: fixed reference frame
- B: moving rigid body
- P: point ***moving*** on B
- $\hat{P}$: point ***fixed*** on B that ***coincides*** with P
- Q: point ***fixed*** on B

The ***velocity*** of P can be found by ***differentiating*** its position vector as follows.

$$
\begin{aligned}
{}^{R}\underset{\sim}{v}_P &= \frac{{}^{R}d\underset{\sim}{p}}{dt} = \frac{{}^{R}d}{dt}\left(\underset{\sim}{q} + \underset{\sim}{r}\right) \\
&= \frac{{}^{R}d\underset{\sim}{q}}{dt} + \frac{{}^{R}d\underset{\sim}{r}}{dt} \\
&= {}^{R}\underset{\sim}{v}_Q + \frac{{}^{B}d\underset{\sim}{r}}{dt} + \left({}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right) \\
&= {}^{R}\underset{\sim}{v}_Q + {}^{B}\underset{\sim}{v}_P + \left({}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)
\end{aligned}
$$

Now, letting $\underset{\sim}{r} \to \underset{\sim}{0}$ (that is, letting Q be $\hat{P}$) the desired result is obtained. The ***acceleration*** of P can be found by ***differentiating*** the expression for its velocity.

$$
\begin{aligned}
{}^{R}\underset{\sim}{a}_P &= \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_Q + {}^{B}\underset{\sim}{v}_P + \left({}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right) \\
&= \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_Q\right) + \left\{\frac{{}^{R}d}{dt}\left({}^{B}\underset{\sim}{v}_P\right)\right\} + \left\{\frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right\} \\
&= {}^{R}\underset{\sim}{a}_Q + \left\{\frac{{}^{B}d}{dt}\left({}^{B}\underset{\sim}{v}_P\right) + \left({}^{R}\underset{\sim}{\omega}_B \times {}^{B}\underset{\sim}{v}_P\right)\right\} + \left\{\left({}^{R}\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_B \times \left(\frac{{}^{B}d\underset{\sim}{r}}{dt} + \left({}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right)\right\} \\
&= {}^{R}\underset{\sim}{a}_Q + \left\{{}^{B}\underset{\sim}{a}_P + \left({}^{R}\underset{\sim}{\omega}_B \times {}^{B}\underset{\sim}{v}_P\right)\right\} + \left\{\left({}^{R}\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + \left({}^{R}\underset{\sim}{\omega}_B \times {}^{B}\underset{\sim}{v}_P\right) + {}^{R}\underset{\sim}{\omega}_B \times \left({}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right\}
\end{aligned}
$$

Now, letting $\underset{\sim}{r} \to \underset{\sim}{0}$ (that is, letting Q be $\hat{P}$) the desired result is obtained.