

Introductory Control Systems

Solutions to Linear, Ordinary Differential Equations Using Laplace Transforms

- *Linear ordinary differential equations* (ODE's) may be solved using *Laplace transforms*. There are *three* steps in the solution process.
 1. The *Laplace transform* is used to *convert* the *differential equation* into an *algebraic equation*.
 2. The algebraic equation is *solved* for the Laplace transform of the solution, and the concept of *partial fractions* is used to *separate* the solution into its elementary components.
 3. Finally, the *inverse Laplace transform* is taken of each of the elementary components to find the solution in the time domain.
- The *definitions* of the Laplace transform and the inverse Laplace transform are given below on page 2. Fortunately, however, *tables* of *known transforms* can be used to solve many differential equations.
- For most cases, the *properties* of the Laplace transform and *tables* of transforms of common functions are used to find solutions. A list of helpful properties of Laplace transforms is given below.

Properties of the Laplace Transform and Inverse Laplace Transforms

1. $\mathcal{L}(kf(t)) = k\mathcal{L}(f(t)) = kF(s)$ and $\mathcal{L}^{-1}(kF(s)) = kf(t)$
2. $\mathcal{L}(f_1(t) \pm f_2(t)) = F_1(s) \pm F_2(s)$ and $\mathcal{L}^{-1}(F_1(s) \pm F_2(s)) = f_1(t) \pm f_2(t)$
3. $\mathcal{L}\left(\frac{df}{dt}\right) = sF(s) - f(0)$ and $\mathcal{L}\left(\frac{d^2f}{dt^2}\right) = s^2F(s) - sf(0) - \frac{df}{dt}(0)$
4. $\mathcal{L}\left(\int_0^t f(t)dt\right) = \frac{F(s)}{s}$
5. Initial Value Theorem: $\lim_{t \rightarrow 0} (f(t)) = \lim_{s \rightarrow \infty} (sF(s))$ (if the time limit exists)
6. Final Value Theorem: $\lim_{t \rightarrow \infty} (f(t)) = \lim_{s \rightarrow 0} (sF(s))$ (if the time limit exists)

Note: Item 3 shows the formulae for the Laplace transforms of the first two derivatives of a function. There is a more general formula for the Laplace transform of the n^{th} derivative of a function that is used for differential equations of order three or higher.

Definitions of the Laplace Transform and Inverse Laplace Transform

The Laplace transform of a function $f(t)$ is defined as

$$\mathcal{L}(f(t)) = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

and the inverse Laplace transform is defined to be

$$\mathcal{L}^{-1}(F(s)) = f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st} ds$$

where s is a complex-valued variable in the Laplace domain.