

## Intermediate Dynamics

### Example: Slider on a Rotating Bar

**Problem:** Given the position coordinates  $(x, \phi, \theta)$ , find

${}^R \underline{v}_P$  the velocity of  $P$  and  ${}^R \underline{a}_P$  the acceleration of  $P$ .

**Solution:**

Reference frames:  $\begin{cases} D: (\underline{e}_1, \underline{e}_2, \underline{e}_3) \\ B: (\underline{e}_r, \underline{e}_\theta, \underline{e}_3) \end{cases}$

Angular velocity of  $B$ :

$$\begin{aligned} {}^R \underline{\omega}_B &= {}^R \underline{\omega}_D + {}^D \underline{\omega}_B \\ &= \Omega \underline{e}_2 + \omega \underline{e}_3 \end{aligned}$$

Angular acceleration of  $B$ :

$${}^R \underline{\alpha}_B = \frac{{}^R d}{dt} ({}^R \underline{\omega}_B) = \dot{\Omega} \underline{e}_2 + \dot{\omega} \underline{e}_3 + \omega \dot{\underline{e}}_3 = \dot{\Omega} \underline{e}_2 + \dot{\omega} \underline{e}_3 + \omega (\Omega \underline{e}_2 \times \underline{e}_3)$$

$${}^R \underline{\alpha}_B = \omega \Omega \underline{e}_1 + \dot{\Omega} \underline{e}_2 + \dot{\omega} \underline{e}_3$$

Velocity of  $P$ :

$${}^R \underline{v}_P = {}^R \underline{v}_{\hat{P}} + {}^B \underline{v}_P \quad (P \text{ moves on } B \text{ and } \hat{P} \text{ is fixed on } B)$$

Here,

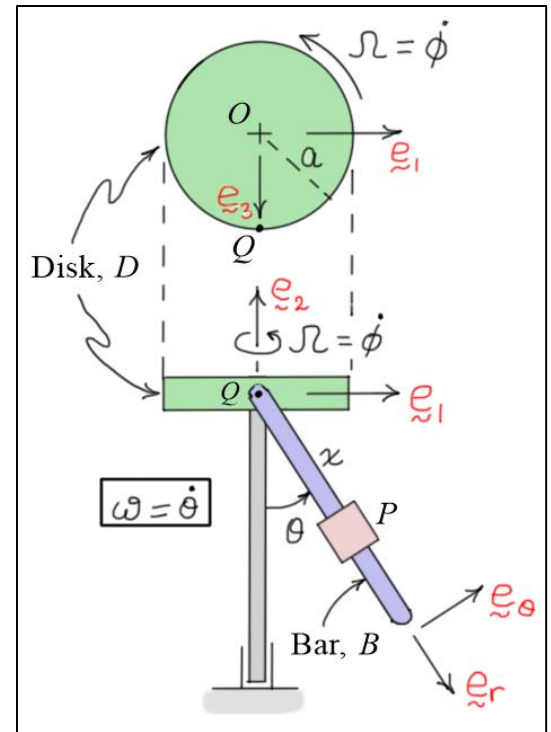
$${}^R \underline{v}_{\hat{P}} = {}^R \underline{v}_Q + {}^R \underline{v}_{\hat{P}/Q} = a \Omega \underline{e}_1 + ({}^R \underline{\omega}_B \times \underline{r}_{\hat{P}/Q}) = a \Omega \underline{e}_1 + \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 0 & \Omega & \omega \\ x S_\theta & -x C_\theta & 0 \end{vmatrix}$$

$$\Rightarrow {}^R \underline{v}_{\hat{P}} = (a \Omega + x \omega C_\theta) \underline{e}_1 + (x \omega S_\theta) \underline{e}_2 - (x \Omega S_\theta) \underline{e}_3$$

$${}^B \underline{v}_P = \dot{x} \underline{e}_r = \dot{x} (S_\theta \underline{e}_1 - C_\theta \underline{e}_2)$$

Combining the above results gives the result for  ${}^R \underline{v}_P$ .

$${}^R \underline{v}_P = (a \Omega + x \omega C_\theta + \dot{x} S_\theta) \underline{e}_1 + (x \omega S_\theta - \dot{x} C_\theta) \underline{e}_2 - (x \Omega S_\theta) \underline{e}_3$$



Acceleration of  $P$ :

$$\boxed{{}^R \underline{a}_P = {}^R \underline{a}_{\hat{P}} + {}^B \underline{a}_P + 2({}^R \underline{\omega}_B \times {}^B \underline{v}_P)}$$

Here,

$${}^R \underline{a}_{\hat{P}} = {}^R \underline{a}_Q + {}^R \underline{a}_{\hat{P}/Q}$$

$$\boxed{{}^R \underline{a}_Q = a \dot{\Omega} \underline{e}_1 - a \Omega^2 \underline{e}_3}$$

$${}^R \underline{a}_{\hat{P}/Q} = ({}^R \underline{\alpha}_B \times \underline{r}_{\hat{P}/Q}) + ({}^R \underline{\omega}_B \times {}^R \underline{v}_{\hat{P}/Q}) = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \omega \Omega & \dot{\Omega} & \dot{\omega} \\ x S_\theta & -x C_\theta & 0 \end{vmatrix} + \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 0 & \Omega & \omega \\ x \omega C_\theta & x \omega S_\theta & -x \Omega S_\theta \end{vmatrix}$$

$$\boxed{{}^R \underline{a}_{\hat{P}/Q} = (x \dot{\omega} C_\theta - x \Omega^2 S_\theta - x \omega^2 S_\theta) \underline{e}_1 + (x \dot{\omega} S_\theta + x \omega^2 C_\theta) \underline{e}_2 - (x \omega \Omega C_\theta + x \dot{\Omega} S_\theta + x \omega \Omega C_\theta) \underline{e}_3}$$

$$\boxed{{}^B \underline{a}_P = \ddot{x} \underline{e}_r = \ddot{x} (S_\theta \underline{e}_1 - C_\theta \underline{e}_2)}$$

$$2 {}^R \underline{\omega}_B \times {}^B \underline{v}_P = 2 \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 0 & \Omega & \omega \\ \dot{x} S_\theta & -\dot{x} C_\theta & 0 \end{vmatrix}$$

$$\boxed{2 {}^R \underline{\omega}_B \times {}^B \underline{v}_P = 2 \left[ (\dot{x} \omega C_\theta) \underline{e}_1 + (\dot{x} \omega S_\theta) \underline{e}_2 - (\dot{x} \Omega S_\theta) \underline{e}_3 \right]}$$

Combining all the above terms gives the result for  ${}^R \underline{a}_P$ .

$$\boxed{{}^R \underline{a}_P = (a \dot{\Omega} + x \dot{\omega} C_\theta - x S_\theta (\omega^2 + \Omega^2) + \ddot{x} S_\theta + 2 \dot{x} \omega C_\theta) \underline{e}_1 + (x \dot{\omega} S_\theta + x \omega^2 C_\theta - \ddot{x} C_\theta + 2 \dot{x} \omega S_\theta) \underline{e}_2 - (a \Omega^2 + x \dot{\Omega} S_\theta + 2 x \omega \Omega C_\theta + 2 \dot{x} \Omega S_\theta) \underline{e}_3}$$