Multibody Dynamics
Euler Parameters and Angular Velocity Components

In earlier notes relationships between angular velocity components and various sets of orientation angles and their derivatives were derived. Here, a similar ***relationship*** between ***angular velocity components*** and the ***Euler parameters*** and ***their derivatives*** is derived.

Angular Velocity Components in a Fixed Frame

Previously, it was found that the time derivative of a transformation matrix may be written in terms of angular ***velocity components*** in a ***fixed frame*** as follows.

$$\boxed{[\dot{R}]^T = [\tilde{\omega}][R]^T} \tag{1}$$

Post-multiplying the above equation by $[R]$ gives

$$\boxed{[\dot{R}]^T[R] = [\tilde{\omega}][R]^T[R] = [\tilde{\omega}]} \qquad \text{or} \qquad \boxed{[\tilde{\omega}] = [\dot{R}]^T[R]} \tag{2}$$

Recall that the elements of $[\tilde{\omega}]$ are the ***components*** of ${}^R\underset{\sim}{\omega}_B$ in the ***fixed frame*** $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$.

Using Eq. (2), equations relating the individual angular velocity components and the Euler parameters, and the time derivatives of the Euler parameters can be derived. This can be done by noting that

$$\boxed{\omega_1 = \tilde{\omega}_{32} = \sum_{i=1}^{3} \dot{R}^T_{3i} R_{i2}} \quad \boxed{\omega_2 = \tilde{\omega}_{13} = \sum_{i=1}^{3} \dot{R}^T_{1i} R_{i3}} \quad \boxed{\omega_3 = \tilde{\omega}_{21} = \sum_{i=1}^{3} \dot{R}^T_{2i} R_{i1}} \tag{3}$$

Substituting expressions into the above equations for $\dot{R}^T_{ij}$ and R_{ij} in terms of the Euler parameters, it can be shown that

$$\boxed{\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{Bmatrix} = 2 \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & -\varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & -\varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & -\varepsilon_3 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{bmatrix} \begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix}} \tag{4}$$

Note that the last equation is simply the derivative of the Euler parameter constraint equation, $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$.

Eq. (4) may be written in the more compact form

$$\boxed{\{\omega\} = 2[E]\{\dot{\varepsilon}\}} \tag{5}$$

where the definition of the matrix $[E]$ is obvious by comparing Eqs. (4) and (5). It can be shown that $[E]$ is an ***orthogonal*** matrix. Hence, the above equation can be ***easily inverted***.

$$\boxed{\{\dot{\varepsilon}\} = \tfrac{1}{2}[E]^T\{\omega\}} \tag{6}$$

These equations are ***analogous*** to those that relate the ***angular velocity components*** to the derivatives of a set of ***orientation angles***. Recall, however, that in those equations, singularities always exist. The above equation exhibits ***no singularities***.

<u>Angular Velocity Components in the Body-Fixed Frame</u>

Previously, it was found that the time derivative of a transformation matrix may be written in terms of ***angular velocity components*** in the ***body frame*** as follows.

$$\boxed{[\dot{R}]^T = [R]^T[\tilde{\omega}']} \tag{7}$$

Pre-multiplying the above equation by $[R]$ gives

$$\boxed{[R][\dot{R}]^T = [R][R]^T[\tilde{\omega}'] = [\tilde{\omega}']} \quad \text{or} \quad \boxed{[\tilde{\omega}'] = [R][\dot{R}]^T} \tag{8}$$

Recall that elements of $[\tilde{\omega}']$ are ***components*** of ${}^R\underset{\sim}{\omega}_B$ in the ***body-fixed frame*** $B:\left(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\right)$. Using this relationship, equations relating the ***angular velocity components*** in the ***body-fixed reference frame*** and the ***Euler parameters*** and ***their time derivatives*** can be derived. This can be done by noting

$$\boxed{\omega_1' = \tilde{\omega}_{32} = \sum_{i=1}^{3} R_{3i}\dot{R}_{i2}^T} \quad \boxed{\omega_2' = \tilde{\omega}_{13} = \sum_{i=1}^{3} R_{1i}\dot{R}_{i3}^T} \quad \boxed{\omega_3' = \tilde{\omega}_{21} = \sum_{i=1}^{3} R_{2i}\dot{R}_{i1}^T} \tag{9}$$

Substituting expressions into the above equations for $\dot{R}_{ij}^T$ and R_{ij} in terms of the Euler parameters, it can be shown that

$$\boxed{\begin{Bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \\ 0 \end{Bmatrix} = 2\begin{bmatrix} \varepsilon_4 & \varepsilon_3 & -\varepsilon_2 & -\varepsilon_1 \\ -\varepsilon_3 & \varepsilon_4 & \varepsilon_1 & -\varepsilon_2 \\ \varepsilon_2 & -\varepsilon_1 & \varepsilon_4 & -\varepsilon_3 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{bmatrix}\begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix}} \tag{10}$$

Note that (as before) the last equation is simply the derivative of the Euler parameter constraint equation, $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$.

Eq. (10) can be written in the more compact form

$$\boxed{\{\omega'\} = 2[E']\{\dot{\varepsilon}\}} \tag{11}$$

where the definition of the matrix $[E']$ is obvious by comparing Eqs. (10) and (11). It can be shown that $[E']$ is an ***orthogonal*** matrix. Hence, Eq. (11) can be ***easily inverted*** to give

$$\boxed{\{\dot{\varepsilon}\} = \tfrac{1}{2}[E']^T\{\omega'\}} \tag{12}$$

As noted above, this matrix equation exhibits ***no singularities***.

Calculation of First Angular Velocity Components of Equations (4) and (10)

To show how Eqs. (4) and (10) can be derived, the first component of each is derived here. Using the first of Eqs. (3),

$$\begin{aligned}
\omega_1 &= \dot{R}_{31}^T R_{12} + \dot{R}_{32}^T R_{22} + \dot{R}_{33}^T R_{32} \\
&= \tfrac{d}{dt}\left[2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4)\right]\left[2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4)\right] + \tfrac{d}{dt}\left[2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4)\right]\left[(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2)\right] \\
&\quad + \tfrac{d}{dt}\left[(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2)\right]\left[2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4)\right] \\
&= \left[2(\dot{\varepsilon}_1\varepsilon_3 + \varepsilon_1\dot{\varepsilon}_3 - \dot{\varepsilon}_2\varepsilon_4 - \varepsilon_2\dot{\varepsilon}_4)\right]\left[2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4)\right] \\
&\quad + \left[2(\dot{\varepsilon}_2\varepsilon_3 + \varepsilon_2\dot{\varepsilon}_3 + \dot{\varepsilon}_1\varepsilon_4 + \varepsilon_1\dot{\varepsilon}_4)\right]\left[(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2)\right] \\
&\quad + \left[2(-\varepsilon_1\dot{\varepsilon}_1 - \varepsilon_2\dot{\varepsilon}_2 + \varepsilon_3\dot{\varepsilon}_3 + \varepsilon_4\dot{\varepsilon}_4)\right]\left[2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4)\right] \\
&= 4\left[(\cancel{\varepsilon_1\varepsilon_2\varepsilon_3} + \varepsilon_3^2\varepsilon_4)\dot{\varepsilon}_1 - (\cancel{\varepsilon_1\varepsilon_2\varepsilon_4} + \varepsilon_3\varepsilon_4^2)\dot{\varepsilon}_2 + (\varepsilon_1^2\varepsilon_2 + \cancel{\varepsilon_1\varepsilon_3\varepsilon_4})\dot{\varepsilon}_3 - (\varepsilon_1\varepsilon_2^2 + \cancel{\varepsilon_2\varepsilon_3\varepsilon_4})\dot{\varepsilon}_4\right] \\
&\quad + \left[2(\dot{\varepsilon}_2\varepsilon_3 + \varepsilon_2\dot{\varepsilon}_3 + \dot{\varepsilon}_1\varepsilon_4 + \varepsilon_1\dot{\varepsilon}_4)\right]\left[(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2)\right] \\
&\quad + 4\left[(\varepsilon_1^2\varepsilon_4 - \cancel{\varepsilon_1\varepsilon_2\varepsilon_3})\dot{\varepsilon}_1 + (\cancel{\varepsilon_1\varepsilon_2\varepsilon_4} - \varepsilon_2^2\varepsilon_3)\dot{\varepsilon}_2 + (\varepsilon_3^2\varepsilon_2 - \cancel{\varepsilon_1\varepsilon_3\varepsilon_4})\dot{\varepsilon}_3 + (\cancel{\varepsilon_2\varepsilon_3\varepsilon_4} - \varepsilon_1\varepsilon_4^2)\dot{\varepsilon}_4\right]
\end{aligned}$$

Cancelling the terms noted in the previous equation gives the following.

$$\begin{aligned}
\omega_1 &= 4\left[(\varepsilon_3^2\varepsilon_4)\dot{\varepsilon}_1 - (\varepsilon_3\varepsilon_4^2)\dot{\varepsilon}_2 + (\varepsilon_1^2\varepsilon_2)\dot{\varepsilon}_3 - (\varepsilon_1\varepsilon_2^2)\dot{\varepsilon}_4\right] \\
&\quad + \left[2(\dot{\varepsilon}_2\varepsilon_3 + \varepsilon_2\dot{\varepsilon}_3 + \dot{\varepsilon}_1\varepsilon_4 + \varepsilon_1\dot{\varepsilon}_4)\right]\left[(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2)\right] \\
&\quad + 4\left[(\varepsilon_1^2\varepsilon_4)\dot{\varepsilon}_1 + (-\varepsilon_2^2\varepsilon_3)\dot{\varepsilon}_2 + (\varepsilon_3^2\varepsilon_2)\dot{\varepsilon}_3 + (-\varepsilon_1\varepsilon_4^2)\dot{\varepsilon}_4\right]
\end{aligned}$$

Rearranging like terms associated with the derivative of each of the Euler parameters, simplifying, and using the Euler parameter constraint equation $\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2=1\right)$ gives

$$
\begin{aligned}
\omega_1 = {} & 4\left[(\varepsilon_3^2\varepsilon_4)\dot{\varepsilon}_1-(\varepsilon_3\varepsilon_4^2)\dot{\varepsilon}_2+(\varepsilon_1^2\varepsilon_2)\dot{\varepsilon}_3-(\varepsilon_1\varepsilon_2^2)\dot{\varepsilon}_4\right] \\
& +\left[2(\dot{\varepsilon}_2\varepsilon_3+\varepsilon_2\dot{\varepsilon}_3+\dot{\varepsilon}_1\varepsilon_4+\varepsilon_1\dot{\varepsilon}_4)\right]\left[(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)\right] \\
& +4\left[(\varepsilon_1^2\varepsilon_4)\dot{\varepsilon}_1+(-\varepsilon_2^2\varepsilon_3)\dot{\varepsilon}_2+(\varepsilon_3^2\varepsilon_2)\dot{\varepsilon}_3+(-\varepsilon_1\varepsilon_4^2)\dot{\varepsilon}_4\right] \\
= {} & 2\varepsilon_4\left[2\varepsilon_3^2+(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)+2\varepsilon_1^2\right]\dot{\varepsilon}_1+2\varepsilon_3\left[-2\varepsilon_4^2+(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)-2\varepsilon_2^2\right]\dot{\varepsilon}_2 \\
& +2\varepsilon_2\left[2\varepsilon_1^2+(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)+2\varepsilon_3^2\right]\dot{\varepsilon}_3+2\varepsilon_1\left[-2\varepsilon_2^2+(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)-2\varepsilon_4^2\right]\dot{\varepsilon}_4 \\
= {} & 2\varepsilon_4\left[\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right]\dot{\varepsilon}_1-2\varepsilon_3\left[\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right]\dot{\varepsilon}_2 \\
& +2\varepsilon_2\left[\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right]\dot{\varepsilon}_3-2\varepsilon_1\left[\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right]\dot{\varepsilon}_4 \\
= {} & 2\left[\varepsilon_4\dot{\varepsilon}_1-\varepsilon_3\dot{\varepsilon}_2+\varepsilon_2\dot{\varepsilon}_3-\varepsilon_1\dot{\varepsilon}_4\right]
\end{aligned}
$$

This result matches the equation in the first row of Eq. (4).

To check the first row of Eq. (10), use the first of Eqs. (9) gives

$$
\begin{aligned}
\omega_1' = {} & R_{31}\dot{R}_{12}^T+R_{32}\dot{R}_{22}^T+R_{33}\dot{R}_{32}^T \\
= {} & \tfrac{d}{dt}\left[2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4)\right]\left[2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)\right]+\tfrac{d}{dt}\left[(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)\right]\left[2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4)\right] \\
& +\tfrac{d}{dt}\left[2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)\right]\left[(-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)\right] \\
= {} & \left[2(\dot{\varepsilon}_1\varepsilon_2+\varepsilon_1\dot{\varepsilon}_2-\dot{\varepsilon}_3\varepsilon_4-\varepsilon_3\dot{\varepsilon}_4)\right]\left[2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)\right] \\
& +\left[2(-\varepsilon_1\dot{\varepsilon}_1+\varepsilon_2\dot{\varepsilon}_2-\varepsilon_3\dot{\varepsilon}_3+\varepsilon_4\dot{\varepsilon}_4)\right]\left[2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4)\right] \\
& +\left[2(\dot{\varepsilon}_2\varepsilon_3+\varepsilon_2\dot{\varepsilon}_3+\dot{\varepsilon}_1\varepsilon_4+\varepsilon_1\dot{\varepsilon}_4)\right]\left[(-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)\right]
\end{aligned}
$$

$$
\begin{aligned}
= {} & 4\left[(\cancel{\varepsilon_1\varepsilon_2\varepsilon_3}+\varepsilon_2^2\varepsilon_4)\dot{\varepsilon}_1+(\varepsilon_1^2\varepsilon_3+\cancel{\varepsilon_1\varepsilon_2\varepsilon_4})\dot{\varepsilon}_2-(\cancel{\varepsilon_1\varepsilon_3\varepsilon_4}+\varepsilon_2\varepsilon_4^2)\dot{\varepsilon}_3-(\varepsilon_1\varepsilon_3^2+\cancel{\varepsilon_2\varepsilon_3\varepsilon_4})\dot{\varepsilon}_4\right] \\
& +4\left[(\varepsilon_1^2\varepsilon_4-\cancel{\varepsilon_1\varepsilon_2\varepsilon_3})\dot{\varepsilon}_1+(\varepsilon_2^2\varepsilon_3-\cancel{\varepsilon_1\varepsilon_2\varepsilon_4})\dot{\varepsilon}_2+(\cancel{\varepsilon_1\varepsilon_3\varepsilon_4}-\varepsilon_2\varepsilon_3^2)\dot{\varepsilon}_3+(\cancel{\varepsilon_2\varepsilon_3\varepsilon_4}-\varepsilon_1\varepsilon_4^2)\dot{\varepsilon}_4\right] \\
& +\left[2(\dot{\varepsilon}_2\varepsilon_3+\varepsilon_2\dot{\varepsilon}_3+\dot{\varepsilon}_1\varepsilon_4+\varepsilon_1\dot{\varepsilon}_4)\right]\left[(-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)\right]
\end{aligned}
$$

Cancelling the terms noted in the previous equation gives

$$
\begin{aligned}
\omega_1' = 4&\left[(\varepsilon_2^2\varepsilon_4)\dot{\varepsilon}_1 + (\varepsilon_1^2\varepsilon_3)\dot{\varepsilon}_2 - (\varepsilon_2\varepsilon_4^2)\dot{\varepsilon}_3 - (\varepsilon_1\varepsilon_3^2)\dot{\varepsilon}_4 \right] \\
&+4\left[(\varepsilon_1^2\varepsilon_4)\dot{\varepsilon}_1 + (\varepsilon_2^2\varepsilon_3)\dot{\varepsilon}_2 + (-\varepsilon_2\varepsilon_3^2)\dot{\varepsilon}_3 + (-\varepsilon_1\varepsilon_4^2)\dot{\varepsilon}_4 \right] \\
&+\left[2(\dot{\varepsilon}_2\varepsilon_3 + \varepsilon_2\dot{\varepsilon}_3 + \dot{\varepsilon}_1\varepsilon_4 + \varepsilon_1\dot{\varepsilon}_4) \right]\left[(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) \right]
\end{aligned}
$$

Rearranging like terms associated with the derivative of each of the Euler parameters, simplifying, and using the Euler parameter constraint equation $\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1\right)$ gives

$$
\begin{aligned}
\omega_1' = 4&\left[(\varepsilon_2^2\varepsilon_4)\dot{\varepsilon}_1 + (\varepsilon_1^2\varepsilon_3)\dot{\varepsilon}_2 - (\varepsilon_2\varepsilon_4^2)\dot{\varepsilon}_3 - (\varepsilon_1\varepsilon_3^2)\dot{\varepsilon}_4 \right] \\
&+4\left[(\varepsilon_1^2\varepsilon_4)\dot{\varepsilon}_1 + (\varepsilon_2^2\varepsilon_3)\dot{\varepsilon}_2 + (-\varepsilon_2\varepsilon_3^2)\dot{\varepsilon}_3 + (-\varepsilon_1\varepsilon_4^2)\dot{\varepsilon}_4 \right] \\
&+\left[2(\dot{\varepsilon}_2\varepsilon_3 + \varepsilon_2\dot{\varepsilon}_3 + \dot{\varepsilon}_1\varepsilon_4 + \varepsilon_1\dot{\varepsilon}_4) \right]\left[(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) \right] \\
&= 2\varepsilon_4\left[2\varepsilon_2^2 + 2\varepsilon_1^2 + (-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) \right]\dot{\varepsilon}_1 + 2\varepsilon_3\left[2\varepsilon_1^2 + 2\varepsilon_2^2 + (-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) \right]\dot{\varepsilon}_2 \\
&\quad -2\varepsilon_2\left[2\varepsilon_4^2 + 2\varepsilon_3^2 + (\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 - \varepsilon_4^2) \right]\dot{\varepsilon}_3 - 2\varepsilon_1\left[2\varepsilon_3^2 + 2\varepsilon_4^2 + (\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 - \varepsilon_4^2) \right]\dot{\varepsilon}_4 \\
&= 2\varepsilon_4\left[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 \right]\dot{\varepsilon}_1 + 2\varepsilon_3\left[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 \right]\dot{\varepsilon}_2 \\
&\quad -2\varepsilon_2\left[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 \right]\dot{\varepsilon}_3 - 2\varepsilon_1\left[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 \right]\dot{\varepsilon}_4 \\
&= 2\left[\varepsilon_4\dot{\varepsilon}_1 + \varepsilon_3\dot{\varepsilon}_2 - \varepsilon_2\dot{\varepsilon}_3 - \varepsilon_1\dot{\varepsilon}_4 \right]
\end{aligned}
$$

This result matches the equation in the first row of Eq. (10).

Note: Clearly proving these equations is somewhat tedious. Use of a numerical symbolic mathematics package can be useful in expediting the work.