Introductory Control Systems
Examples: Using Laplace Transforms to Solve Differential Equations

<u>Examples</u>

1. Unforced Spring-Mass-Damper

   Problem: Solve the differential equation of motion $m\ddot{x} + c\dot{x} + kx = 0$ subject to the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.

   Solution: Taking Laplace transforms of both sides of the differential equation gives

   $$m[s^2 X(s) - s x_0 - \dot{x}_0] + c[sX(s) - x_0] + kX(s) = 0$$

   or

   $$[ms^2 + cs + k]X(s) = [ms + c]x_0 + m\dot{x}_0$$

   Solving for $X(s)$ gives

   $$\boxed{X(s) = \frac{[ms + c]x_0 + m\dot{x}_0}{ms^2 + cs + k} = \left(\frac{[ms + c]x_0}{ms^2 + cs + k}\right) + \left(\frac{m\dot{x}_0}{ms^2 + cs + k}\right)}$$

   Notes:

   a) The two terms on the right side of this equation represent the response of the system to the ***initial position*** and ***initial velocity***, respectively.
   b) The ***characteristic equation*** of the system is found by setting the ***denominator*** of the right side of the equation to ***zero*** (i.e. $ms^2 + cs + k = 0$).
   c) The ***poles*** of the system are the roots of the ***denominator***, and the ***zeros*** of the system are the roots of the ***numerator***.

   Case 1: $k/m = 2$; $c/m = 3$; $x_0 \neq 0$; $\dot{x}_0 = 0$

   Substituting these values into the equation for $X(s)$ gives

   $$\boxed{X(s) = \frac{[ms + c]x_0}{ms^2 + cs + k} = \frac{x_0(s + 3)}{s^2 + 3s + 2} = \frac{x_0(s + 3)}{(s + 1)(s + 2)}} \quad \text{(2 real, unequal poles)}$$

   The solution to the differential equation may be found by taking the ***inverse Laplace transform*** of $X(s)$. Using #8 from the Laplace transform table with $\alpha = 3$, $a = 1$, and $b = 2$ gives

$$x(t)=\mathcal{L}^{-1}(X(s))=\mathcal{L}^{-1}\left[\frac{x_0(s+3)}{(s+1)(s+2)}\right]=x_0\left(2e^{-t}-e^{-2t}\right)$$

Check:

$$\begin{aligned} x(0) &= x_0(2e^0-e^0)=x_0(2-1)=x_0 \\ \dot{x}(0) &= \dot{x}(t)\big|_{t=0}=x_0(-2e^{-t}+2e^{-2t})\big|_{t=0}=x_0(-2+2)=0 \end{aligned}$$

<u>Case 2</u>: $k/m=2$; $c/m=2$; $x_0\neq 0$; $\dot{x}_0=0$

Substituting these values into the equation for $X(s)$ gives

$$X(s)=\frac{[ms+c]x_0}{ms^2+cs+k}=\frac{x_0(s+2)}{s^2+2s+2}=\frac{x_0(s+2)}{(s+1)^2+1}$$ (2 complex conjugate poles)

The solution to the differential equation may be found by taking the ***inverse Laplace transform*** of $X(s)$. Using #18 from the Laplace transform table with $\alpha=2$, $a=1$, and $\omega=1$ gives

$$x(t)=\mathcal{L}^{-1}(X(s))=\mathcal{L}^{-1}\left[\frac{x_0(s+2)}{(s+1)^2+1}\right]=\sqrt{2}\,x_0\,e^{-t}\sin(t+\phi)$$

where $\phi=\tan^{-1}(1/(2-1))=\begin{cases} 0.7854\text{ (rad)}=45\text{ (deg)} \\ 0.7854+\pi\text{ (rad)}=225\text{ (deg)} \end{cases}$.

Check: (using $\phi=0.7854$ (rad))

$$\begin{aligned} x(0) &= x(t)\big|_{t=0}=\sqrt{2}\,x_0\,e^0\sin(0.7854)=\sqrt{2}\,x_0\left(\sqrt{2}/2\right)=x_0 \\ \dot{x}(0) &= \dot{x}(t)\big|_{t=0}=\sqrt{2}\,x_0\left[-e^{-t}\sin(t+0.7854)+e^{-t}\cos(t+0.7854)\right]\Big|_{t=0} \\ &= \sqrt{2}\,x_0\left(-\sqrt{2}/2+\sqrt{2}/2\right)=0 \end{aligned}$$

Note that $\phi=0.7854+\pi$ (rad) ***does not*** satisfy the initial conditions.

2. Spring-Mass-Damper with a *Unit Step Input*

<u>Problem</u>: Solve the differential equation of motion $m\ddot{x}+c\dot{x}+kx=R(t)=u_s(t)$ where $u_s(t)$ is the ***unit step function***. Find the ***final value*** of $x(t)$ using the ***final value theorem***.

Case 1: $m=1;\ \ k/m=2;\ \ c/m=3;\ \ x_0=\dot{x}_0=0$

Taking Laplace transforms of both sides of the differential equation gives

$$\left(ms^2+cs+k\right)X(s)=\mathcal{L}\left(u_s(t)\right)=\tfrac{1}{s} \qquad \text{or} \qquad \boxed{X(s)=\frac{1}{s(s+1)(s+2)}}$$

Using #9 from the Laplace transform tables with $a=1$ and $b=2$ gives

$$\boxed{x(t)=\mathcal{L}^{-1}\left(X(s)\right)=\tfrac{1}{2}\left[1-2e^{-t}+e^{-2t}\right]} \qquad \text{(forced response)}$$

Using the final value theorem, we have $x_{ss}=\lim\limits_{s\to 0}\left(sX(s)\right)=\lim\limits_{s\to 0}\left(\frac{\cancel{s}}{\cancel{s}(s+1)(s+2)}\right)=\frac{1}{2}$.

Case 2: $m=1;\ \ k/m=2;\ \ c/m=3;\ \ x_0\neq 0;\ \ \dot{x}_0=0$

Taking Laplace transforms of both sides of the differential equation gives

$$m[s^2X(s)-sx_0-\cancel{\dot{x}_0}]+c[sX(s)-x_0]+kX(s)=\tfrac{1}{s}$$

or

$$[ms^2+cs+k]X(s)=\tfrac{1}{s}+[ms+c]x_0$$

Solving for $X(s)$ gives

$$\boxed{X(s)=\left(\frac{1}{s(ms^2+cs+k)}\right)+\left(\frac{x_0(ms+c)}{ms^2+cs+k}\right)=\underbrace{\left(\frac{1}{s(s^2+3s+2)}\right)}_{\text{forced response}}+\underbrace{\left(\frac{x_0(s+3)}{(s^2+3s+2)}\right)}_{\text{response due to initial condition}}}$$

Using #8 and #9 from the Laplace transform tables gives

$$\boxed{x(t)=\mathcal{L}^{-1}\left(X(s)\right)=\underbrace{\tfrac{1}{2}\left[1-2e^{-t}+e^{-2t}\right]}_{\text{forced response}}+\underbrace{x_0\left[2e^{-t}-e^{-2t}\right]}_{\text{response due to initial condition}}}$$

Question: What part of this response is ***transient response*** and what part is ***steady-state response***?