Introductory Motion and Control
PID Control of a Spring-Mass-Damper Position (Root Locus Analysis)

Fig. 1 shows a spring-mass-damper (SMD) system with a ***force actuator*** for ***position control***. The spring has stiffness k, the damper has coefficient c, the block has mass m, and the position of the mass is measured by the variable x. The transfer function of the SMD with an actuating force F_a as input and the position x as output is

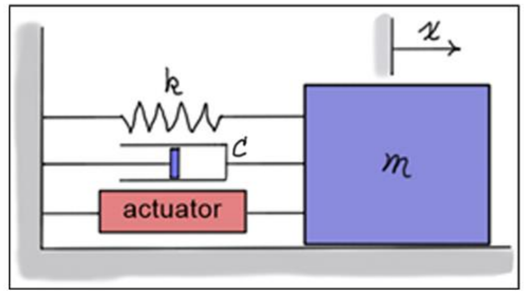
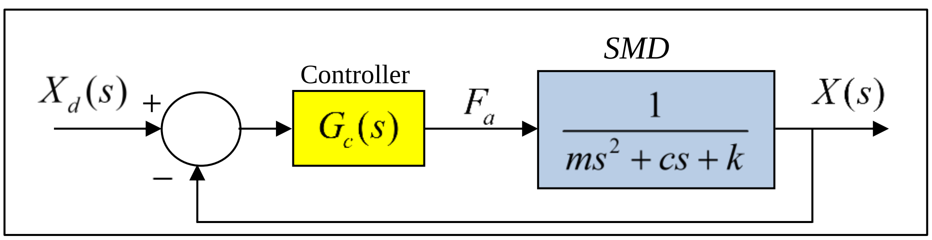


Fig. 1. Spring-Mass-Damper System with Force Actuator

$$\boxed{\frac{X}{F_a}(s)=\frac{1}{ms^2+cs+k}} \qquad (1)$$

Assuming ***ideal*** actuator and sensor responses, the closed-loop position control of the SMD can be described using the following block diagram. Here, X_d represents the ***desired position***, and $G_c(s)$ represents the ***transfer function*** of the controller.



For the analyses that follow, it is assumed the SMD ***parameters*** are: $m=1$ slug, $c=8.8$ (lb-s/ft), and $k=40$ (lb/ft). This represents an ***under-damped, second-order*** plant with

Natural Frequency: $\boxed{\omega_n=\sqrt{40}=6.325\text{ (rad/s)}\approx 1\text{ (Hz)}}$

Damping Ratio: $\boxed{\zeta=\frac{8.8}{2\sqrt{40}}=0.696\approx 0.7}$

<u>Proportional Control</u>

For ***proportional control***, $\boxed{G_c(s)=K}$, and the loop and closed-loop transfer functions are

$$\boxed{GH(s)=\frac{K}{s^2+8.8s+40}} \qquad \boxed{\frac{X}{X_d}(s)=\frac{K}{s^2+8.8s+(40+K)}} \qquad (2)$$

Using $GH(s)$, the RL diagram for the closed-loop system for $K \geq 0$ is shown in ***Fig. 2***. Note that as the value of K is ***increased***, the complex, closed-loop poles move straight up/down, indicating that the ***natural frequency*** is ***increased*** and the ***damping ratio*** is ***decreased*** as K is increased.

This is a ***type-zero*** system and hence will have a ***finite steady-state error*** for a step input.

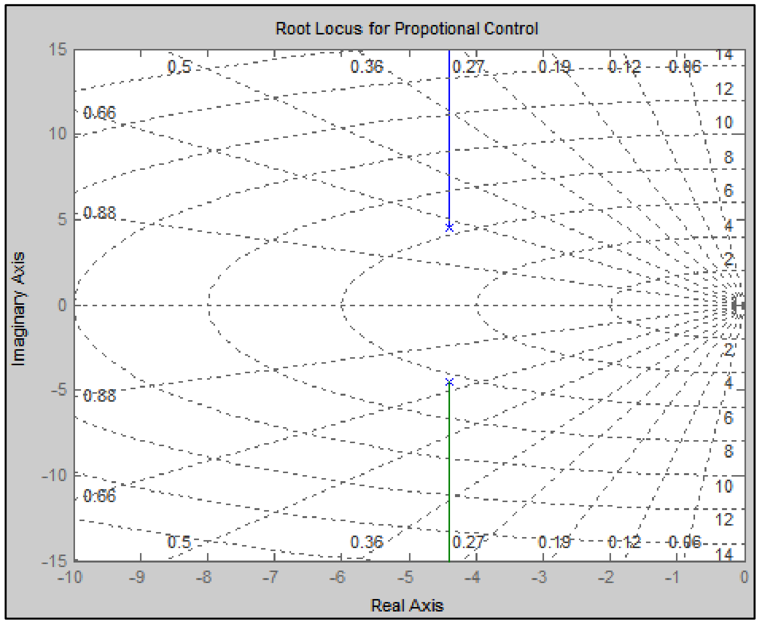


Fig. 2. RL Diagram for Proportional Control

Using the final-value theorem and the closed-loop transfer function, x_{ss} the final value of $x(t)$ to a unit step command is

$$x_{ss} = \lim_{s\to 0}\left(\cancel{s}\cdot\frac{1}{\cancel{s}}\cdot\frac{K}{s^2+8.8s+(40+K)}\right)=\frac{K}{40+K}<1 \tag{3}$$

Eq. (3) indicates that ***large values*** of K ***lead to small steady-state errors***; however, they also lead to ***faster, less damped responses***.

This conclusion is verified in ***Fig. 3*** which shows the closed-loop step responses for proportional control gains K of 100, 500, and 2000. Clearly, it is not possible to achieve low steady-state error and good transient response using only proportional control. To remove the steady-state error and have better response, integral and/or derivative terms must be included.

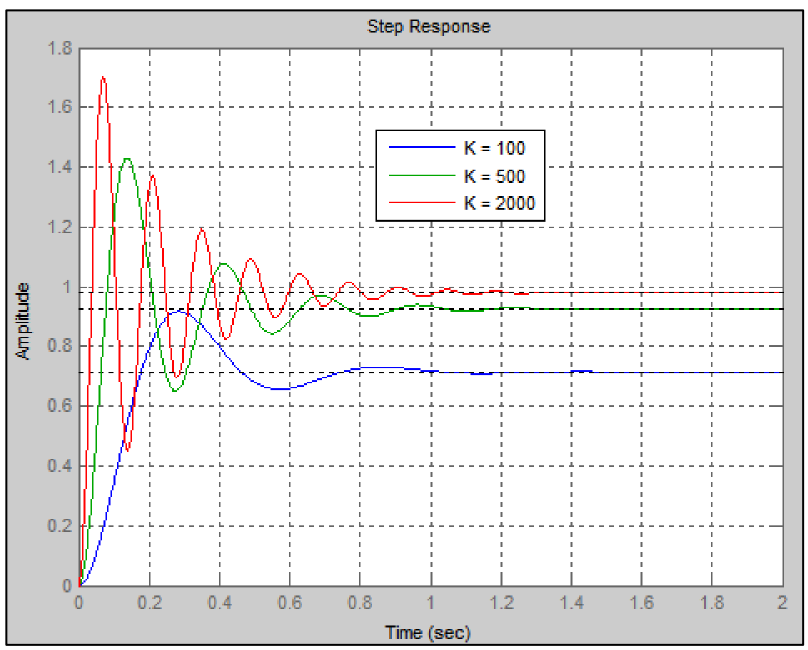


Fig. 3. Step Response for Various Proportional Control Gains

Proportional-Integral (PI) Control

For ***proportional-integral (PI) control***

$$G_c(s) = K_p + \frac{K_I}{s} = \frac{K_p(s+a)}{s} \tag{4}$$

Here, the coefficients K_P and K_I represent the ***proportional*** and ***integral gains***, and the coefficient $a = K_I / K_P$ is the ***ratio*** of the integral and proportional gains. In this case, the loop and closed-loop transfer functions are

$$GH(s) = \frac{K_P(s+a)}{s(s^2+8.8s+40)} \qquad \frac{X}{X_d}(s) = \frac{K_P(s+a)}{s(s^2+8.8s+40)+K_P(s+a)} \tag{5}$$

Using ***integral control*** makes the system ***type-one***, so the ***steady-state error*** due to a step input is ***zero***. This can be verified using the final value theorem to show that $x_{ss} = 1$ when the input is a unit step function. ***Fig. 4*** shows the RL diagram for the closed-loop system with $a = 3$. It also shows the location of the closed-loop poles for a proportional gain $K_P \approx 50$. ***Fig. 5*** shows the ***closed-loop step response*** for $a = 3$ and $K_p = 25$, 50, and 75.

Integral control has ***removed*** the ***steady-state error*** and ***improved*** the ***transient response***, but it has also ***increased*** the system's ***settling time***.

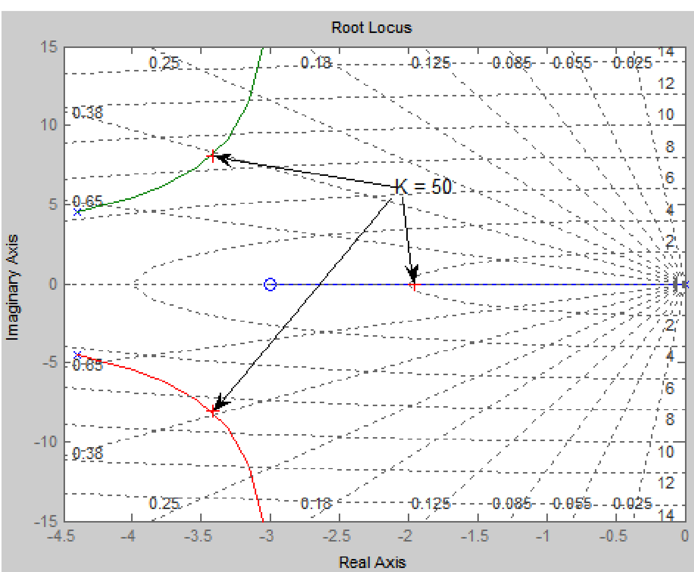


Fig. 4. Root Locus Diagram for PI Control $(a = 3)$

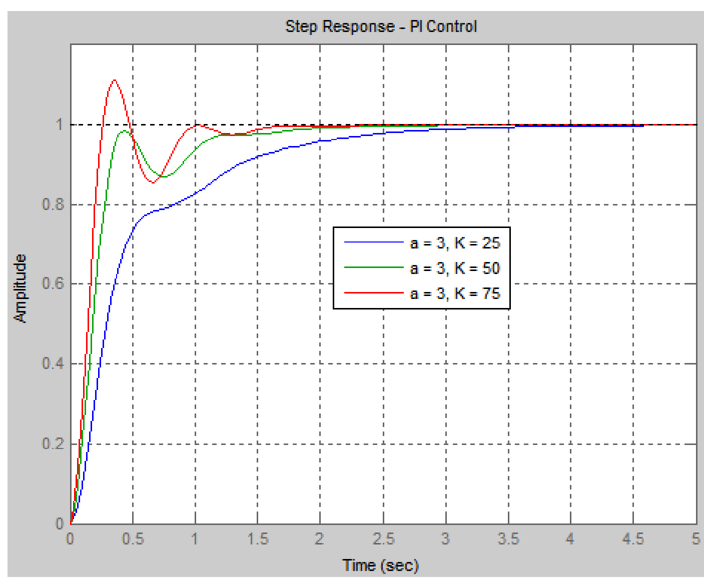


Fig. 5. Step Response for PI Control $(a = 3)$ for Various Proportional Gains

Proportional-Derivative (PD) Control

For ***proportional-derivative (PD) control***: $\boxed{G_c(s) = K_p + K_D s = K_D(s+a)}$ (6)

The coefficients K_P and K_D represent the ***proportional*** and ***derivative gains***, and $a = K_P / K_D$ is the ratio of the proportional and derivative gains. The loop and closed-loop transfer functions are

$$\boxed{GH(s) = \frac{K_D(s+a)}{s^2 + 8.8s + 40}} \qquad \boxed{\frac{X}{X_d}(s) = \frac{K_D(s+a)}{(s^2 + 8.8s + 40) + K_D(s+a)}} \tag{7}$$

Without integral control, this is a ***type-zero*** system, and hence will have a ***finite steady-state error*** to a unit step input. Using the final-value theorem and the closed-loop transfer function, x_{ss} the final value of $x(t)$ to a unit step command is

$$\boxed{x_{ss} = \lim_{s\to 0}\left(\$ \cdot \frac{1}{\$} \cdot \frac{K_D(s+a)}{(s^2 + 8.8s + 40) + K_D(s+a)} \right) = \frac{K_D a}{40 + K_D a} = \frac{K_P}{40 + K_P} < 1} \tag{8}$$

As with proportional control, ***larger proportional gains*** produce ***smaller steady-state errors***. ***Fig. 6*** shows the RL diagram for the closed-loop system with $a = 10$. It also shows the location of the closed-loop poles for $K_D \approx 25.6$. ***Fig. 7*** shows the ***closed-loop step response*** for $a = 10$ and derivative gains of $K_D = 10$, 27, 50, and 75. The PD controller has ***decreased*** the system's ***settling time*** considerably; however, to control the steady-state error, the derivative gain K_D must be high. This ***decreases*** the ***response times*** of the system and can make it ***susceptible*** to ***noise***.

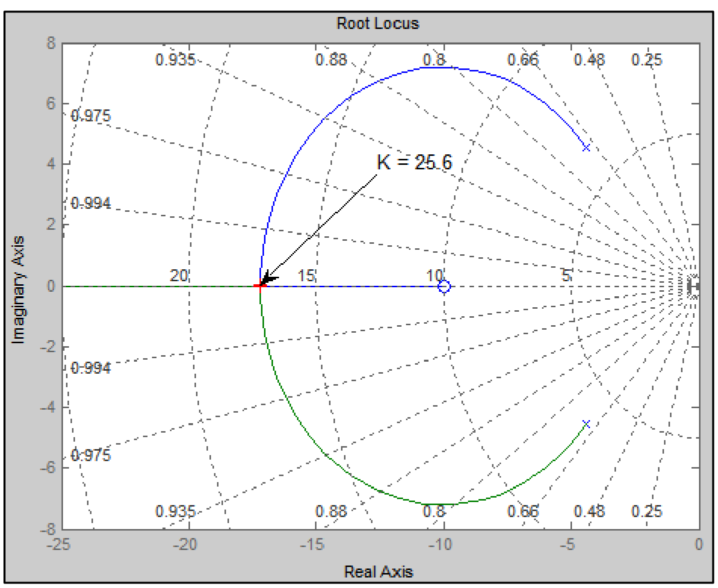


Fig. 6. Root Locus Diagram for PD Control $(a = 10)$

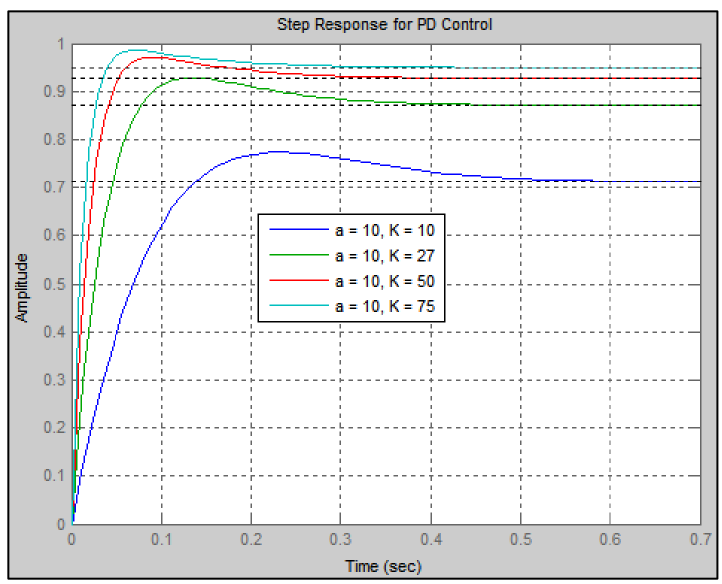


Figure 7. Step Response for PD Control $(a = 10)$ for Various Derivative Gains

Proportional-Integral-Derivative Control

For ***proportional-integral-derivative (PID) control***

$$G_c(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_D(s^2 + as + b)}{s} \qquad (9)$$

The coefficients K_P, K_I, and K_D represent the ***proportional, integral,*** and ***derivative gains***, $a = K_P/K_D$ is the ***ratio*** of the proportional and derivative gains, and $b = K_I/K_D$ is the ***ratio*** of the integral and derivative gains. In this case, the loop and closed-loop transfer functions are

$$GH(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40)} \qquad \frac{X}{X_d}(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40) + K_D(s^2 + as + b)} \qquad (10)$$

Again, with ***integral control***, the system is ***type-one*** and has ***zero steady-state error*** for a step input. ***Fig. 8*** shows the RL diagram of the closed-loop system for $a = 15$ and $b = 50$. The location of the closed-loop poles for $K_D \approx 15.8$ is also shown. ***Fig. 9*** shows the step response of the closed-loop system for $a = 15$, $b = 50$, and various derivative gains.

The PID controller has ***removed steady-state error*** and ***decreased*** the system's ***settling times*** while maintaining a ***reasonable transient response***.

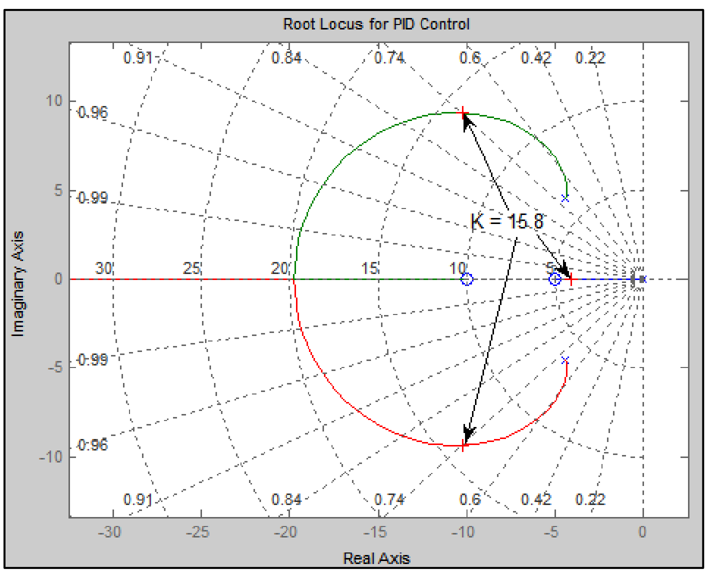


Fig. 8. Root Locus Diagram for PID Control $(a = 15, b = 50)$

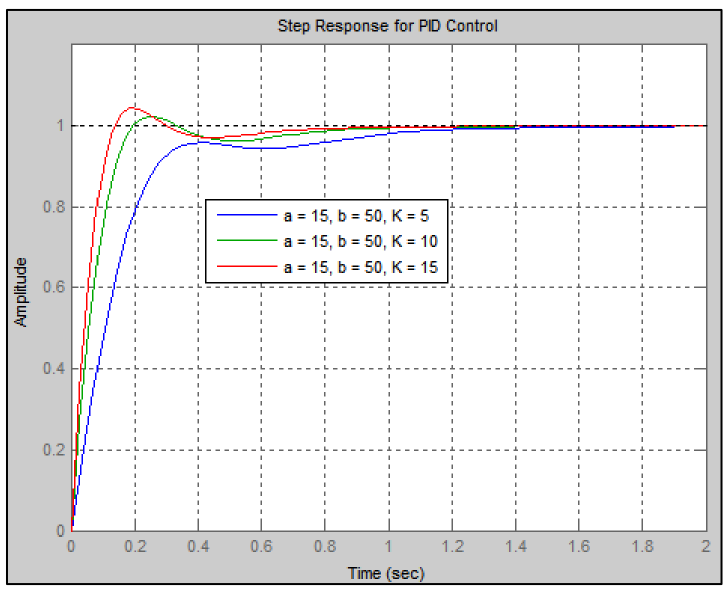


Fig. 9. Step Response for PID Control $(a = 15)$ $(b = 50)$ for Various Derivative Gains