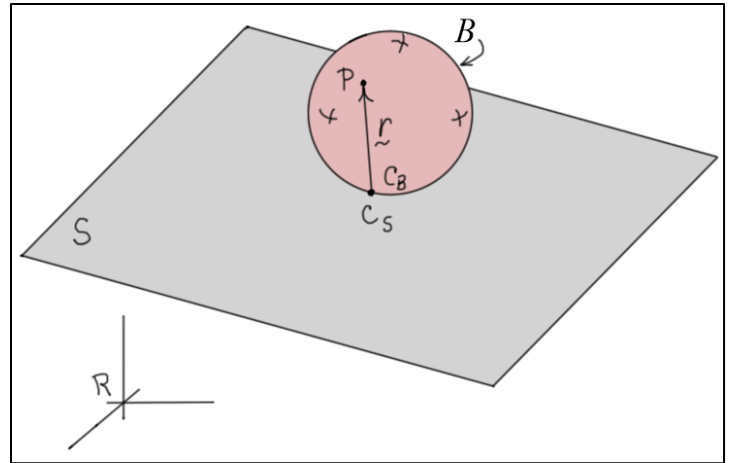


Intermediate Dynamics

Rolling Constraints – Point Contact

When a rigid body rolls (without slipping) on a rigid surface, the surface *constrains* its motion. Consider the body B rolling on the rigid surface S as shown in the diagram. Here,

- S : rigid surface
- B : rigid body
- C_B : contact point on B
- C_S : contact point on S
- R : fixed reference frame



The body B is said to *roll* (without slipping) on S if

$$\boxed{{}^S \underline{v}_{C_B} = \underline{0}} \quad \text{or} \quad \boxed{{}^R \underline{v}_{C_B} = {}^R \underline{v}_{C_S}}$$

The *velocity* of other points of B (e.g. P) can be found by using the formula for *relative velocity*.

$${}^R \underline{v}_P = {}^R \underline{v}_{C_B} + {}^R \underline{v}_{P/C_B}$$

or

$$\boxed{{}^R \underline{v}_P = {}^R \underline{v}_{C_S} + \left({}^R \underline{\omega}_B \times \underline{r}_{P/C_B} \right)}$$

Direct differentiation can then be used to find the acceleration of point P . That is,

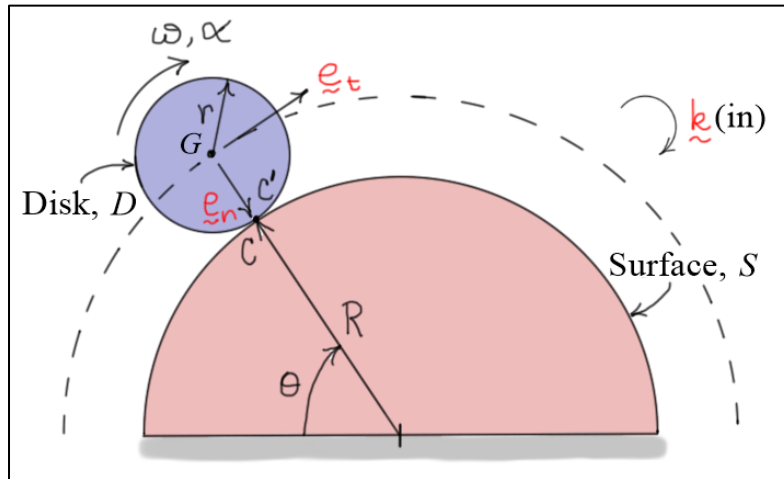
$$\boxed{{}^R \underline{a}_P = \frac{{}^R d}{dt} \left({}^R \underline{v}_P \right)}$$

Note: Even though the velocities of the contact points C_B and C_S are *equal* (i.e. $\underline{v}_{C_B} = \underline{v}_{C_S}$), the accelerations of these points are *not equal* (i.e. $\underline{a}_{C_B} \neq \underline{a}_{C_S}$).

Rolling (without Slipping) in Two Dimensions

Rolling on a Fixed Surface

If a rigid body *rolls* (without slipping) on a *fixed surface*, the point in *contact* with the surface has *zero velocity*. For example, consider a *circular* disk D that rolls on the *circular* surface S as shown below.



Because C' is in *contact* with the point C on the *fixed surface*, its velocity is *zero*. Using this result, the velocity of G (center of the disk) can be calculated using the relative velocity equation.

$$\underline{v}_G = \underline{v}_{C'} + \underline{v}_{G/C'} = \underline{v}_{G/C'} = \underline{\omega}_D \times \underline{r}_{G/C'} = \omega \tilde{k} \times (-r \tilde{e}_n) = r \omega \tilde{e}_t = v \tilde{e}_t$$

or

$$\underline{v}_G = r \omega \tilde{e}_t = v \tilde{e}_t$$

The *acceleration* of G is calculated by *differentiating* \underline{v}_G .

$$\begin{aligned} \underline{a}_G &= \frac{R d}{dt} (r \omega \tilde{e}_t) = r \dot{\omega} \tilde{e}_t + r \omega \dot{\tilde{e}}_t = r \alpha \tilde{e}_t + r \omega (\dot{\theta} \tilde{k} \times \tilde{e}_t) = r \alpha \tilde{e}_t + r \omega \left(\frac{v}{R+r} \tilde{e}_n \right) \\ &= r \alpha \tilde{e}_t + r \omega \left(\frac{r \omega}{R+r} \tilde{e}_n \right) \end{aligned}$$

or

$$\underline{a}_G = r \alpha \tilde{e}_t + \left(\frac{r^2 \omega^2}{R+r} \right) \tilde{e}_n$$

Rolling on a Moving Surface

If a rigid body *rolls* (without slipping) on a *moving surface*, the velocities of the *two contact points*, C and C' , must be the same (i.e. $\underline{v}_{C'} = \underline{v}_C$).

For example, consider a *circular* disk D that rolls on the rotating *circular* surface S as shown in the diagram. As before, the *velocity* of the disk center G can be calculated using the *relative velocity equation*.

$$\begin{aligned}\underline{v}_G &= \underline{v}_{C'} + \underline{v}_{G/C'} = \underline{\omega}_S \times \underline{r}_{C/O} + \underline{\omega}_D \times \underline{r}_{G/C'} \\ &= \Omega \underline{k} \times (-R \underline{e}_n) + \omega \underline{k} \times (-r \underline{e}_n) = (R\Omega + r\omega) \underline{e}_t\end{aligned}$$

or

$$\underline{v}_G = (R\Omega + r\omega) \underline{e}_t = v \underline{e}_t$$

Again, the *acceleration* of G is found by *differentiating* the velocity vector.

$$\begin{aligned}\underline{a}_G &= \frac{d}{dt} (R\Omega + r\omega) \underline{e}_t = (R\dot{\Omega} + r\dot{\omega}) \underline{e}_t + (R\Omega + r\omega) \dot{\underline{e}}_t \\ &= (R\dot{\Omega} + r\dot{\omega}) \underline{e}_t + (R\Omega + r\omega) (\dot{\theta} \underline{k} \times \underline{e}_t) = (R\dot{\Omega} + r\dot{\omega}) \underline{e}_t + (R\Omega + r\omega) \left(\frac{v}{R+r} \underline{e}_n \right)\end{aligned}$$

or

$$\underline{a}_G = (R\dot{\Omega} + r\dot{\omega}) \underline{e}_t + \left(\frac{(R\Omega + r\omega)^2}{R+r} \underline{e}_n \right)$$

Note: Even though the *velocities* of the contact points C and C' are *equal* (i.e. $\underline{v}_{C'} = \underline{v}_C$), the *accelerations* of these points are *not equal* (i.e. $\underline{a}_{C'} \neq \underline{a}_C$). However, in *two dimensions*, it can be shown that the *components* of these *accelerations* along the *tangent direction* (indicated by the unit vector \underline{e}_t) are *equal* (i.e. $\underline{a}_{C'} \cdot \underline{e}_t = \underline{a}_C \cdot \underline{e}_t$).

