

Introductory Control Systems

Examples – Partial Fraction Expansions

1. Given: $X(s) = \frac{5s + 3}{(s + 1)(s + 2)(s + 3)}$

Find: a) the partial fraction expansion of $X(s)$, and b) $x(t)$.

Solution: Given the characteristic equation has **real, unequal roots**, the partial fraction expansion has the form

$$X(s) = \frac{K_1}{s + 1} + \frac{K_2}{s + 2} + \frac{K_3}{s + 3}$$

where the coefficients K_i ($i = 1, 2, 3$) can be calculated as follows.

$$K_1 = [(s + 1)X(s)]_{s=-1} = \left(\frac{5s + 3}{(s + 2)(s + 3)} \right)_{s=-1} = -1$$

$$K_2 = [(s + 2)X(s)]_{s=-2} = \left(\frac{5s + 3}{(s + 1)(s + 3)} \right)_{s=-2} = 7$$

$$K_3 = [(s + 3)X(s)]_{s=-3} = \left(\frac{5s + 3}{(s + 1)(s + 2)} \right)_{s=-3} = -6$$

So, the partial fraction expansion is

$$X(s) = \left(\frac{-1}{s + 1} \right) + \left(\frac{7}{s + 2} \right) + \left(\frac{-6}{s + 3} \right)$$

Using the Laplace transform table gives

$$x(t) = -e^{-t} + 7e^{-2t} - 6e^{-3t}$$

2. Given: $X(s) = \frac{5s + 15}{(s + 2)(s^2 + 3s + 9)}$

Find: a) the partial fraction expansion of $X(s)$, and b) $x(t)$.

Solution: Given the characteristic equation has **one real root** and a **pair of complex conjugate** roots, the partial fraction expansion is of the form

$$X(s) = \left(\frac{K}{s+2} \right) + \left(\frac{As+B}{s^2+3s+9} \right)$$

where the coefficient K may be found as before

$$K = [(s+2)X(s)]_{s=-2} = \left[\frac{5s+15}{s^2+3s+9} \right]_{s=-2} = \frac{5}{7}$$

and the coefficients A and B are found by *clearing fractions* as follows

$$\begin{aligned} 5s+15 &= K(s^2+3s+9) + (As+B)(s+2) \\ &= (K+A)s^2 + (2A+B+3K)s + (9K+2B) \end{aligned}$$

Equating the coefficients of the powers of s on both sides of the equation gives

$$A + K = 0 \quad (s^2)$$

$$2A + B + 3K = 5 \quad (s^1)$$

$$9K + 2B = 15 \quad (s^0)$$

Solving the first and third equations gives $A = -\frac{5}{7}$ and $B = \frac{30}{7}$. So, the partial fraction expansion is

$$X(s) = \frac{5}{7} \left(\frac{1}{s+2} \right) - \frac{5}{7} \left(\frac{s-6}{s^2+3s+9} \right) = \frac{5}{7} \left(\frac{1}{s+2} \right) - \frac{5}{7} \left(\frac{s+(-6)}{(s+\frac{3}{2})^2 + \frac{27}{4}} \right)$$

Using #4 and #18 in the Laplace transform tables gives

$$x(t) = \frac{5}{7} \left[e^{-2t} - 3.055 e^{-1.5t} \sin(2.5981t + \phi) \right] \quad \phi = \tan^{-1} \left(\frac{2.5981}{-6-1.5} \right) = \begin{cases} -0.3335 \text{ (rad)} \\ \text{or} \\ 2.8081 \text{ (rad)} \end{cases}$$

The correct choice of ϕ must be made to *satisfy* the *initial conditions* of the differential equation (not given here).

3. Given:
$$X(s) = \frac{s^5 + a_1s^4 + a_2s^3 + a_3s^2 + a_4s + a_5}{s^6 + 11s^5 + 79s^4 + 427s^3 + 1510s^2 + 3800s + 4000}$$

Find: What is the form of $x(t)$? Identify the *steady-state* and *transient* parts of $x(t)$.

Solution: Using the root solving feature on your calculator, the *poles* of $X(s)$ are found to be $\pm 5j$, $-2 \pm 3.4641j$, -5 , and -2 . So, the partial fraction expansion and $x(t)$ have the forms

$$X(s) = \left(\frac{K_1}{s+2} \right) + \left(\frac{K_2}{s+5} \right) + \left(\frac{A_1s + B_1}{s^2 + 25} \right) + \left(\frac{A_2s + B_2}{s^2 + 4s + 16} \right)$$

$$x(t) = \underbrace{K_1 e^{-2t} + K_2 e^{-5t} + C_1 e^{-2t} \sin(\sqrt{12}t + \phi_1)}_{\text{transient response}} + \underbrace{C_2 \sin(5t + \phi_2)}_{\text{steady-state response}}$$