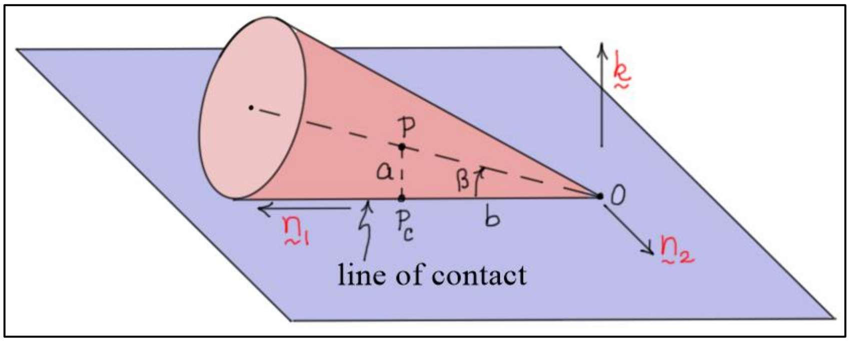
Intermediate Dynamics
Rolling Constraints – Line Contact

Cone Rolling on a Flat Plane

In earlier notes, rolling (without slipping) with a single contact point was discussed. Consider now rolling when there is a ***line of contact*** between the bodies.



For example, consider a ***right circular cone*** *C* rolling on a flat horizontal plane. If the plane is ***fixed***, then all the points on the line of contact of the cone must have ***zero velocity***, and the cone will roll in a ***circular path*** with the point *O* remaining fixed.

To analyze the kinematics of the cone as it rolls, the directions of a reference frame $S:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{k})$ are defined as shown in the diagram. The direction $\underset{\sim}{n}_1$ points along the ***line of contact***, the direction $\underset{\sim}{k}$ is ***normal*** to the plane, and the direction $\underset{\sim}{n}_2 = \underset{\sim}{k} \times \underset{\sim}{n}_1$. Because all the points on the ***contact line*** have ***zero velocity***, the ***angular velocity*** of the cone must be along the $\underset{\sim}{n}_1$ direction. That is,

$$\boxed{{}^R\underset{\sim}{\omega}_C = \omega \underset{\sim}{n}_1}$$

The ***velocity*** of any point of the cone can be calculated using the relative velocity equation. For example, suppose P_C is a ***point*** on the ***contact line***, and *P* is a ***point*** along the cone's ***centerline***, a distance a ***above*** P_C. Then, the ***velocity*** of *P* can be calculated as

$$ {}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{P_C} + {}^R\underset{\sim}{v}_{P/P_C} = \underset{\text{zero}}{{}^R\underset{\sim}{v}_{P_C}} + \left({}^R\underset{\sim}{\omega}_C \times {}^R\underset{\sim}{r}_{P/P_C}\right) = \omega \underset{\sim}{n}_1 \times a\underset{\sim}{k}$$

$$\Rightarrow \boxed{{}^R\underset{\sim}{v}_P = -a\omega \underset{\sim}{n}_2}$$

Note that as points *P* are examined from the vertex of the cone to the center of its base the ***lengths*** a (and, hence, the ***velocities*** of the points *P*) ***increase linearly***. Consequently, the cone rolls in a circular path with the point *O* remaining fixed.

The ***angular acceleration*** of the cone is found by ***differentiating*** ${}^R\underset{\sim}{\omega}_C$.

$$\boxed{{}^R\underset{\sim}{\alpha}_C = \frac{{}^R d}{dt}\left(\omega\underset{\sim}{n}_1\right) = \dot{\omega}\underset{\sim}{n}_1 + \omega\dot{\underset{\sim}{n}}_1 = \dot{\omega}\underset{\sim}{n}_1 + \omega\left({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{n}_1\right) = \dot{\omega}\underset{\sim}{n}_1 + \omega\left(\Omega\,\underset{\sim}{k} \times \underset{\sim}{n}_1\right) = \dot{\omega}\underset{\sim}{n}_1 + \omega\Omega\underset{\sim}{n}_2}$$

Here, ${}^R\underset{\sim}{\omega}_S = \Omega\,\underset{\sim}{k}$ represents the angular velocity of the frame *S* in a fixed frame *R*. The angular rates ω and Ω are ***not independent***, because the cone is ***rolling*** (without slipping).

To find a ***relationship between*** the ***two rates***, consider again a point *P* on the centerline of the cone. As noted above, the velocity of *P* may be written as ${}^R\underset{\sim}{v}_P = -a\,\omega\underset{\sim}{n}_2$. However, it could also be written as ${}^R\underset{\sim}{v}_P = b\Omega\underset{\sim}{n}_2$ where b represents the distance from P_C to *O*. Comparing these two expressions yields the result, $\boxed{\Omega = -\left(a/b\right)\omega = -\omega\tan(\beta)}$, where β represents the ***half angle*** of the cone. Using this result, ${}^R\underset{\sim}{\alpha}_C$ may be written as

$$\boxed{{}^R\underset{\sim}{\alpha}_C = \dot{\omega}\underset{\sim}{n}_1 - \omega^2\tan\left(\beta\right)\underset{\sim}{n}_2}$$

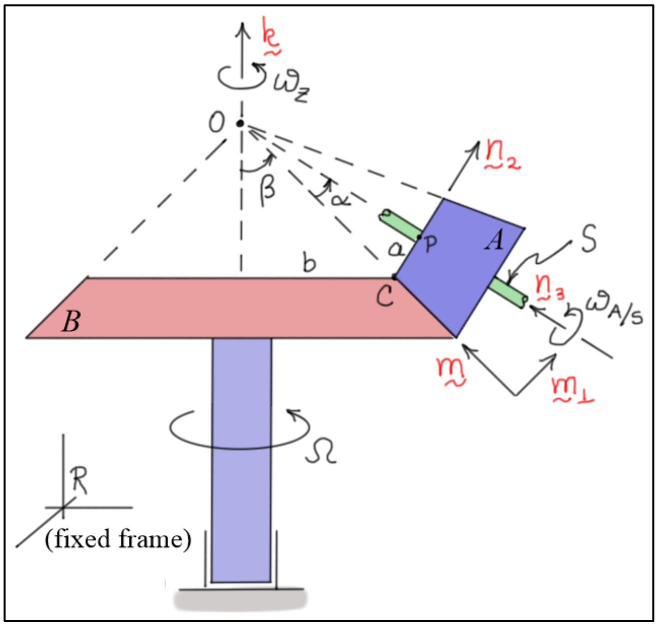
The ***acceleration*** of *P* is found by differentiating the velocity ${}^R\underset{\sim}{v}_P$.

$${}^R\underset{\sim}{a}_P = \frac{{}^R d}{dt}\left(-a\,\omega\underset{\sim}{n}_2\right) = -a\left(\dot{\omega}\underset{\sim}{n}_2 + \omega\dot{\underset{\sim}{n}}_2\right) = -a\left(\dot{\omega}\underset{\sim}{n}_2 + \omega\left(\Omega\,\underset{\sim}{k} \times \underset{\sim}{n}_2\right)\right) = -a\left(\dot{\omega}\underset{\sim}{n}_2 - \omega\Omega\underset{\sim}{n}_1\right)$$

$$\Rightarrow \boxed{{}^R\underset{\sim}{a}_P = -a\,\dot{\omega}\underset{\sim}{n}_2 - a\,\omega^2\tan\left(\beta\right)\underset{\sim}{n}_1}$$

Beveled Gears

Beveled gears are a practical example of bodies that roll with a line of contact. Each gear may be thought of as part of a cone. Two contacting gears may be thought of as ***two cones rolling on each other***. Consider, for example, the system of two beveled gears shown. Gear *B* is affixed to a shaft that rotates with speed Ω about the $\underset{\sim}{k}$ direction. Gear *A* rolls on gear *B* and rotates freely on the axle *S*. The axle rotates at a rate ω_z about the $\underset{\sim}{k}$ direction (pivoting about point *O*), and gear *A* rotates relative to the axle at a rate $\omega_{A/S}$.



To analyze the kinematics of this system, define a unit vector set fixed in the axle as $S:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$. The direction $\underset{\sim}{n}_3$ is ***pointed along*** axle *S* towards *O*, the direction $\underset{\sim}{n}_2$ is ***perpendicular*** to the axle as shown, and the direction $\underset{\sim}{n}_1 = \underset{\sim}{n}_2 \times \underset{\sim}{n}_3$. The angles α and β are the ***half angles*** of the two cones (as shown), and the sum of these two angles is $\gamma \triangleq \alpha + \beta$. Point *C* is on the contact line between the two gears, and point *P* is a point on the axle.

To find a relationship between the angular rates Ω, ω_z, and $\omega_{A/S}$, use the concept of relative velocity. Assuming point *C* is the contact point on gear *A*, and point *P* is the center of that gear, write

$${}^R\underset{\sim}{v}_C = {}^R\underset{\sim}{v}_P + {}^R\underset{\sim}{v}_{C/P}$$

Here,

$${}^R\underset{\sim}{v}_P = -\left(b + aC_\gamma\right)\omega_z \underset{\sim}{n}_1$$

$$\begin{aligned}{}^R\underset{\sim}{v}_{C/P} &= {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{C/P} = \left({}^R\underset{\sim}{\omega}_S + {}^S\underset{\sim}{\omega}_A\right) \times \left(-a\underset{\sim}{n}_2\right) = \left[\omega_z \underset{\sim}{k} + \omega_{A/S}\underset{\sim}{n}_3\right] \times \left(-a\underset{\sim}{n}_2\right) \\ &= \left[\omega_z\left(S_\gamma \underset{\sim}{n}_2 + C_\gamma \underset{\sim}{n}_3\right) + \omega_{A/S}\underset{\sim}{n}_3\right] \times \left(-a\underset{\sim}{n}_2\right) \\ &= a\left(\omega_{A/S} + \omega_z C_\gamma\right)\underset{\sim}{n}_1\end{aligned}$$

Also, considering *C* to be the contact point on gear *B*, the velocity of *C* may be written as ${}^R\underset{\sim}{v}_C = -b\Omega\underset{\sim}{n}_1$. Setting the velocities of the two contact points equal to each other and simplifying gives

$$\boxed{\omega_{A/S} = (b/a)(\omega_z - \Omega)}$$

The Relative Angular Velocities ${}^B\underset{\sim}{\omega}_A$ and ${}^A\underset{\sim}{\omega}_B$

As noted above, when a cone rolls on a fixed plane, the ***angular velocity*** of the cone relative to the plane is ***directed*** along the ***line of contact***. Similarly, when beveled gears roll on each other, the ***relative angular velocities*** ${}^B\underset{\sim}{\omega}_A$ and ${}^A\underset{\sim}{\omega}_B$ must be ***directed*** along the ***line of contact***. Hence,

$$\boxed{{}^B\underset{\sim}{\omega}_A = \omega_{A/B}\underset{\sim}{m} = -{}^A\underset{\sim}{\omega}_B}$$

This result can be verified by finding the angular velocities and using the results found above. For example, using the angular velocity summation rule, write

$$\begin{aligned}
{}^{B}\underset{\sim}{\omega}_A &= {}^{R}\underset{\sim}{\omega}_A - {}^{R}\underset{\sim}{\omega}_B = \left(\omega_z\, \underset{\sim}{k} + \omega_{A/S}\, \underset{\sim}{n}_3\right) - \Omega\underset{\sim}{k} = \left(\omega_z - \Omega\right)\underset{\sim}{k} + \omega_{A/S}\underset{\sim}{n}_3 \\
&= \left(\omega_z - \Omega\right)\left(C_\beta\, \underset{\sim}{m} + S_\beta\, \underset{\sim}{m}_\perp\right) + \tfrac{b}{a}\left(\omega_z - \Omega\right)\left(C_\alpha\, \underset{\sim}{m} - S_\alpha\, \underset{\sim}{m}_\perp\right) \qquad (1) \\
&\Rightarrow \boxed{{}^{B}\underset{\sim}{\omega}_A = \left(\omega_z - \Omega\right)\left((C_\beta + \tfrac{b}{a}C_\alpha)\underset{\sim}{m} + (S_\beta - \tfrac{b}{a}S_\alpha)\underset{\sim}{m}_\perp\right)}
\end{aligned}$$

Here,

$$\frac{b}{a} = \frac{b}{\left|\overline{OC}\right|}\cdot\frac{\left|\overline{OC}\right|}{a} = \frac{S_\beta}{S_\alpha}$$

$$C_\beta + \tfrac{b}{a}C_\alpha = C_\beta + \frac{S_\beta C_\alpha}{S_\alpha} = \frac{C_\beta S_\alpha + S_\beta C_\alpha}{S_\alpha} = \frac{S_{\alpha+\beta}}{S_\alpha}$$

$$S_\beta - \tfrac{b}{a}S_\alpha = S_\beta - \frac{S_\beta S_\alpha}{S_\alpha} = \frac{S_\beta S_\alpha - S_\beta S_\alpha}{S_\alpha} = 0$$

Substituting these results into Eq. (1) gives

$$\boxed{{}^{B}\underset{\sim}{\omega}_A = (\omega_z - \Omega)\left(\frac{S_{\alpha+\beta}}{S_\alpha}\right)\underset{\sim}{m}}$$

So, the relative angular velocity is directed along the line of contact as expected.