**Introductory Motion and Control**
**PID Position Control of a Spring-Mass-Damper: Analysis Summary**

Fig. 1 shows a spring-mass-damper system with a force actuator for position control. The spring has stiffness $k$, the damper has coefficient $c$, the block has mass $m$, and the position of the mass is measured by the variable $x$. The ***transfer function*** of the SMD with the actuating force $F_a$ as input and the position $x$ as output is

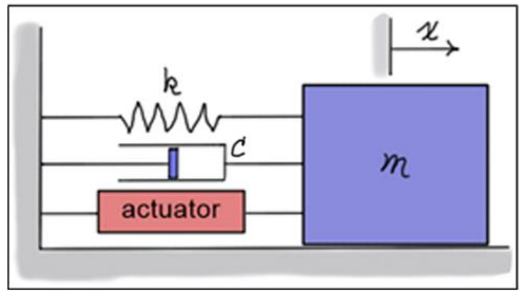
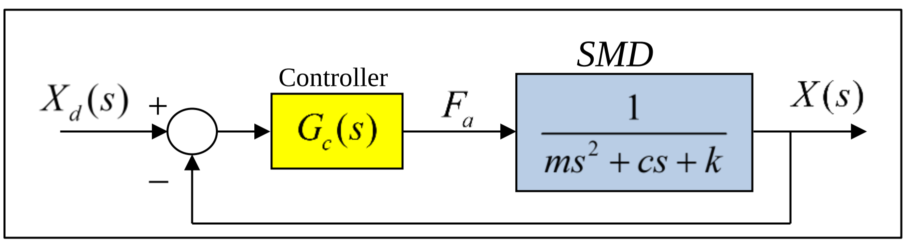



Fig. 1. Spring-Mass-Damper System with Force Actuator

$$\boxed{\frac{X}{F_a}(s) = \frac{1}{ms^2 + cs + k}} \qquad (1)$$

Assuming ideal actuator and sensor responses, the closed-loop position control of the SMD can be described using the following block diagram. Here, $X_d$ represents the ***desired position***, and $G_c(s)$ represents the ***transfer function*** of the controller.




It is assumed here that the SMD parameters are: $m = 1$ slug, $c = 8.8$ (lb-s/ft), and $k = 40$ (lb/ft). This represents an under-damped, second-order plant with the following ***natural frequency*** and ***damping ratio***.

$$\boxed{\begin{aligned} \omega_n &= \sqrt{40} = 6.325 \text{ (rad/s)} \approx 1 \text{ (Hz)} \\ \zeta &= \frac{8.8}{2\sqrt{40}} = 0.696 \approx 0.7 \end{aligned}}$$

<u>Proportional Control</u>

If ***proportional control*** is used, then $\boxed{G_c(s) = K}$, and the ***loop transfer function*** and ***closed-loop transfer functions*** are

$$\boxed{GH(s) = \frac{K}{s^2 + 8.8s + 40}} \qquad \boxed{\frac{X}{X_d}(s) = \frac{K}{s^2 + 8.8s + (40+K)}} \qquad (2)$$

This is a ***type-zero*** system and will have a ***finite steady-state error*** for a step input. Using the final-value theorem and the closed-loop transfer function, $x_{ss}$ the ***final value*** of $x(t)$ to a ***unit step*** command is

$$\boxed{x_{ss} = \lim_{s \to 0} \left( \$ \cdot \frac{1}{\$} \cdot \frac{K}{s^2 + 8.8s + (40+K)} \right) = \frac{K}{40+K} < 1} \tag{3}$$

***Eq. (3)*** indicates that ***large values*** of $K$ ***lead to smaller steady-state errors***; however, as seen below, they also lead to a ***faster, less damped responses***.

The root locus diagram for the closed-loop system for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in ***Fig. 2***. Note that as the value of $K$ is ***increased***, the closed-loop poles move straight up/down, indicating the natural frequency is ***increased*** and the damping ratio is ***decreased***. Also, as the value of $K$ is ***increased***, the ***phase (stability) margin is decreased.***

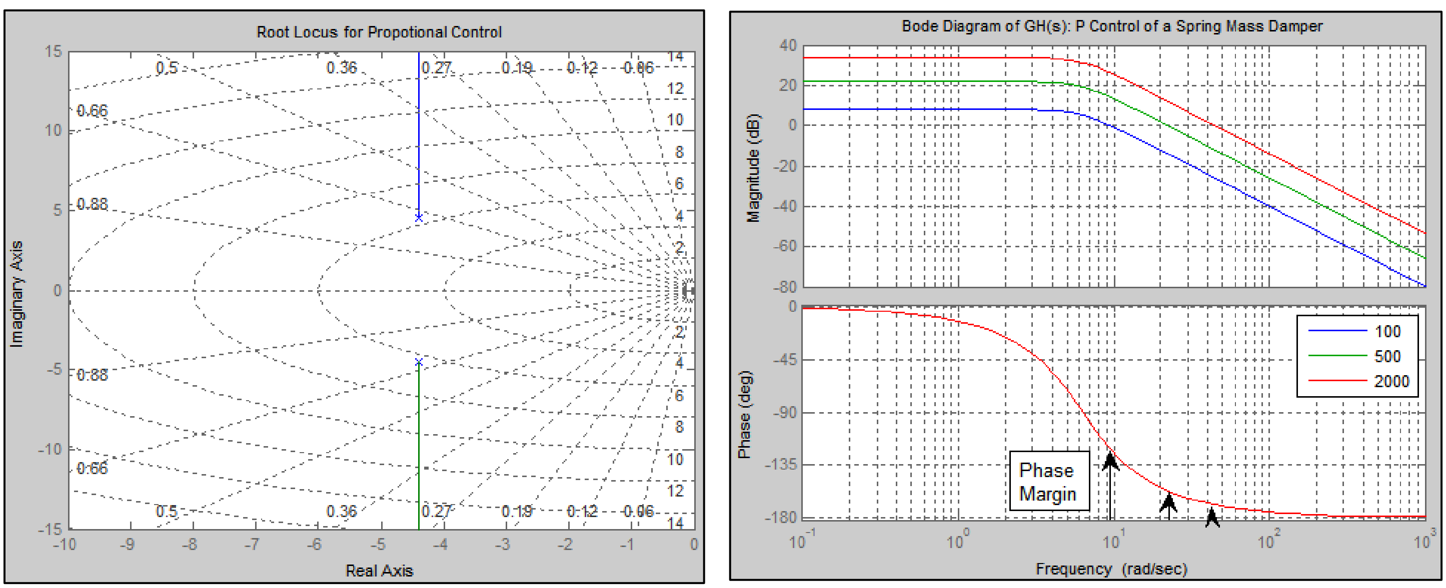



Fig. 2. Root Locus Diagram and Bode Diagram for ( $GH(s)$ ) for Proportional Control

***Fig. 3*** shows step responses and Bode diagrams of the closed-loop system for proportional gains $K$ of 100, 500, and 2000. As the gain is ***increased*** the system response is ***faster*** and ***less damped***. The Bode diagram correspondingly shows ***larger bandwidths*** and ***larger resonant magnitudes***. Clearly, it is not possible to achieve low steady-state error and good transient response using only proportional control. As the gain is increased, the response becomes ***faster***, but it has a ***lower phase margin***. To remove the steady-state error and have better response, integral and/or derivative terms must be included in the controller.

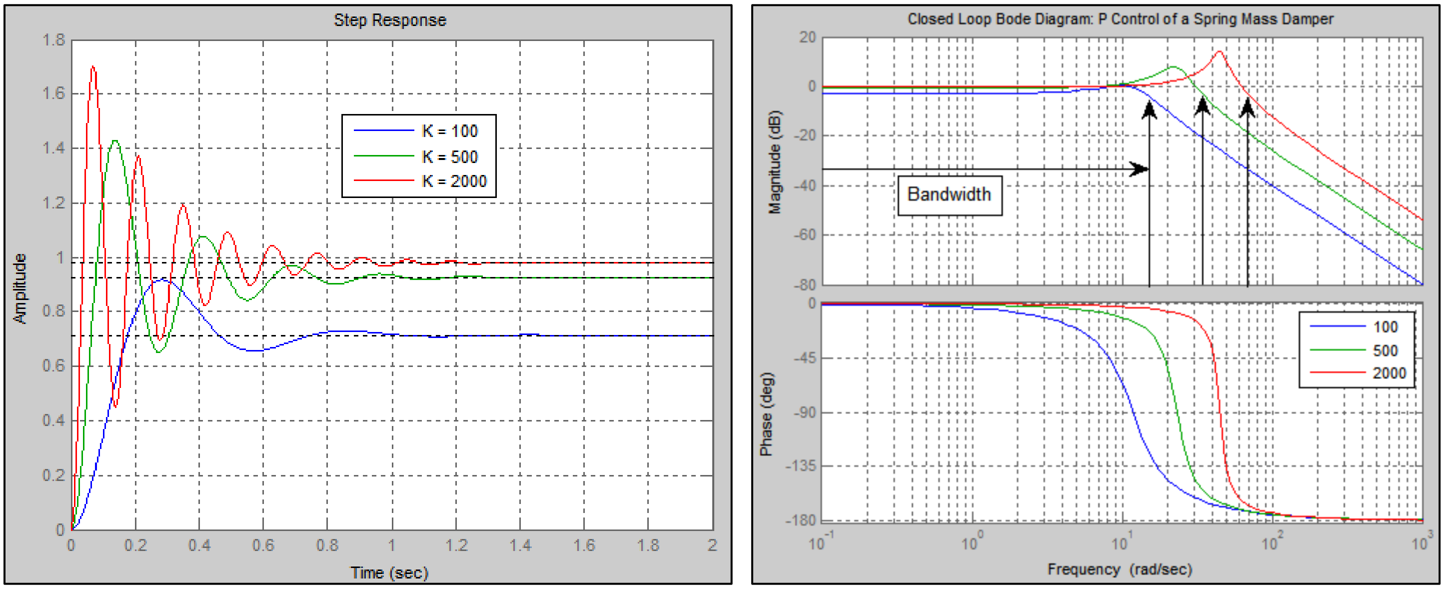



Fig. 3. Closed Loop Step Response and Bode Diagrams for P Control

## Proportional-Integral (PI) Control

If ***proportional-integral (PI) control*** is used, then

$$G_c(s) = K_p + \frac{K_I}{s} = \frac{K_p(s+a)}{s} \tag{4}$$

The parameters $K_P$ and $K_I$ represent the ***proportional*** and ***integral*** gains, and the parameter $a = K_I / K_P$ is the ***ratio*** of the integral and proportional gains. The loop and closed-loop transfer functions for this system are

$$GH(s) = \frac{K_P(s+a)}{s(s^2+8.8s+40)} \qquad \frac{X}{X_d}(s) = \frac{K_P(s+a)}{s(s^2+8.8s+40)+K_P(s+a)} \tag{5}$$

***Integral control*** makes the system a ***type-one*** system, so the ***steady-state error*** due to a step input is ***zero***. This can be verified using the final value theorem to show that $x_{ss} = 1$ for a unit step input.

The root locus diagram for the closed-loop system (with $a = 3$) for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in ***Fig. 4***. The root locus diagram also shows the locations of the closed-loop poles for a proportional gain $K_P = 50$. Note the integral controller has ***added a third, slower pole*** to the system and has ***moved the asymptotes*** of the complex poles closer to the imaginary axis. For low gains, the system is ***slow and stable*** (first order dominant).

As the proportional gain is ***increased***, the system becomes ***faster*** with a ***decreasing phase margin***. The Bode diagram shows that the gain could be increased somewhat above $K_p = 25$ without significantly decreasing the stability margin. However, further increases will decrease the phase margin.

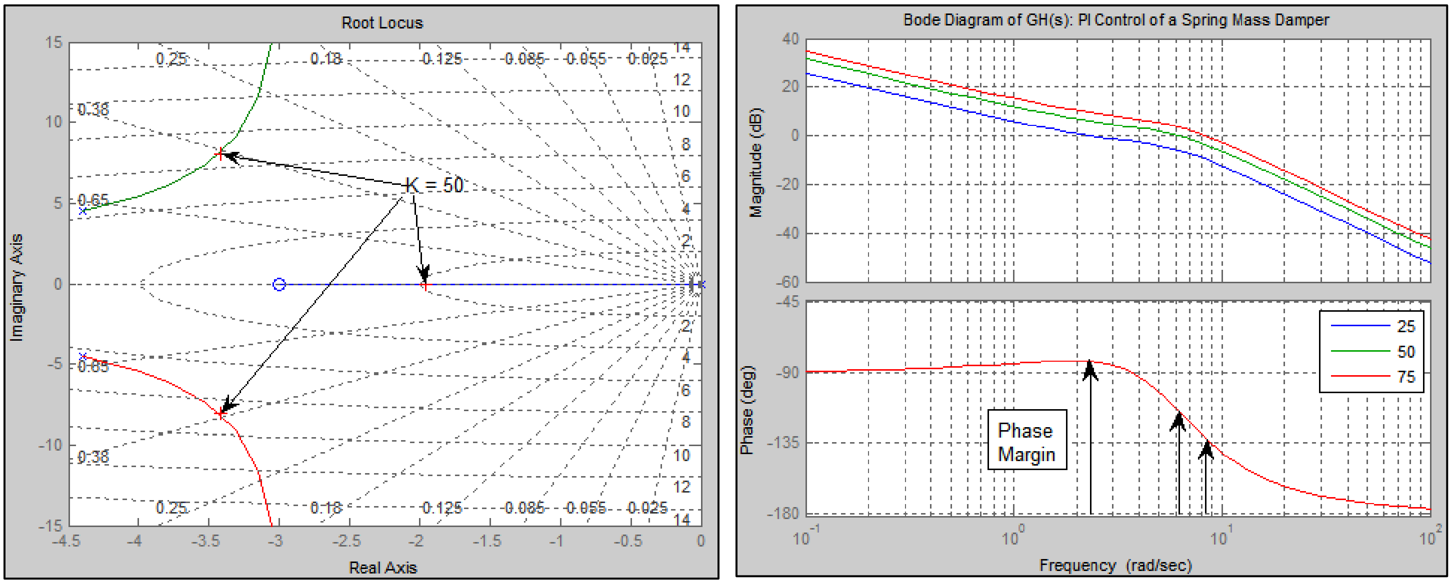



Fig. 4. Root Locus Diagram and Bode Diagram for ( $GH(s)$ ) for PI Control $(a = 3)$

***Fig. 5*** shows step responses and Bode diagrams of the closed loop system for $a = 3$ and proportional gains of $K_p = 25$, 50, and 75. Integral control has ***removed*** the ***steady-state step error*** and ***improved*** the ***transient response***, but it has also ***increased*** the ***system's settling time***. Settling times can be lowered by ***increasing*** the gain. This will ***increase*** the ***system bandwidth***, but it will also ***decrease*** the ***stability margin***.

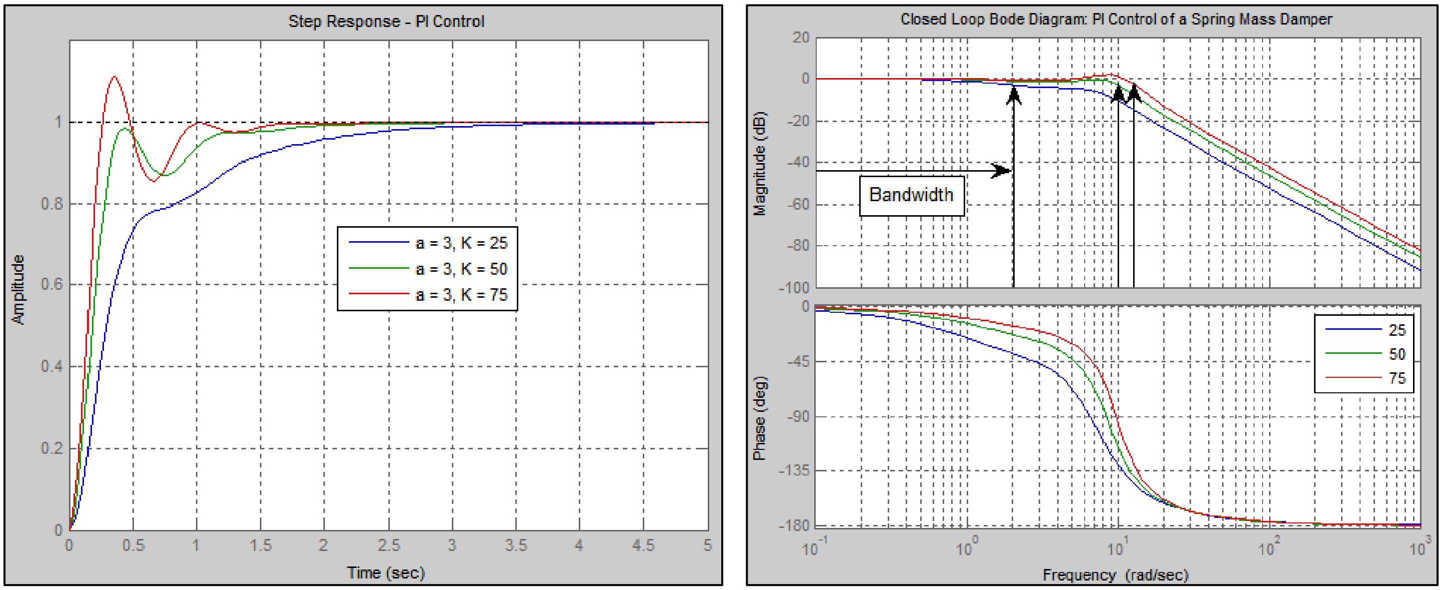



Fig. 5. Closed Loop Step Response and Bode Diagrams for PI Control $(a = 3)$

<u>Proportional-Derivative (PD) Control</u>

If ***proportional-derivative (PD) control*** is used, then

$$\boxed{G_c(s) = K_p + K_D s = K_D(s+a)} \tag{6}$$

The parameters $K_P$ and $K_D$ represent the ***proportional*** and ***derivative*** gains, and the parameter $a = K_P / K_D$ is the ***ratio*** of the proportional and derivative gains. The loop and closed-loop transfer functions for this system are

$$\boxed{GH(s) = \frac{K_D(s+a)}{s^2 + 8.8s + 40}} \qquad \boxed{\frac{X}{X_d}(s) = \frac{K_D(s+a)}{(s^2 + 8.8s + 40) + K_D(s+a)}} \tag{7}$$

Without the integral control, this is again a ***type-zero*** system, and hence will have a ***finite steady-state error*** to a unit step input. Using the final-value theorem and the closed-loop transfer function, $x_{ss}$ the final value of $x(t)$ to a unit step command is

$$x_{ss} = \lim_{s \to 0} \left( \$ \cdot \frac{1}{\$} \cdot \frac{K_D(s+a)}{(s^2 + 8.8s + 40) + K_D(s+a)} \right) = \frac{K_D a}{40 + K_D a} = \frac{K_P}{40 + K_P} < 1 \tag{8}$$

As with simple proportional control, the ***larger the proportional gain***, the ***smaller the steady-state error***.

The root locus diagram for the closed-loop system (with $a = 10$) for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in ***Fig. 6***. The root locus diagram also shows the locations of the closed-loop poles for a derivative gain $K_D \approx 25.6$. As the gain is ***increased*** the system's poles become ***faster*** and ***more damped***. The Bode diagram indicates that the phase margin never drops below 90 degrees indicating a ***very stable*** closed-loop system for any gain.

***Fig. 7*** shows step responses and Bode diagrams of the closed loop system for $a = 10$ and derivative gains of $K_D = 10$, 27, 50, and 75. The PD controller has ***decreased*** the ***system's settling time*** considerably. However, to control the steady-state error, the derivative gain $K_D$ must be high. This will ***decrease*** the ***response times*** and ***increase*** the ***bandwidth*** of the system and may make it ***susceptible*** to ***noise***.

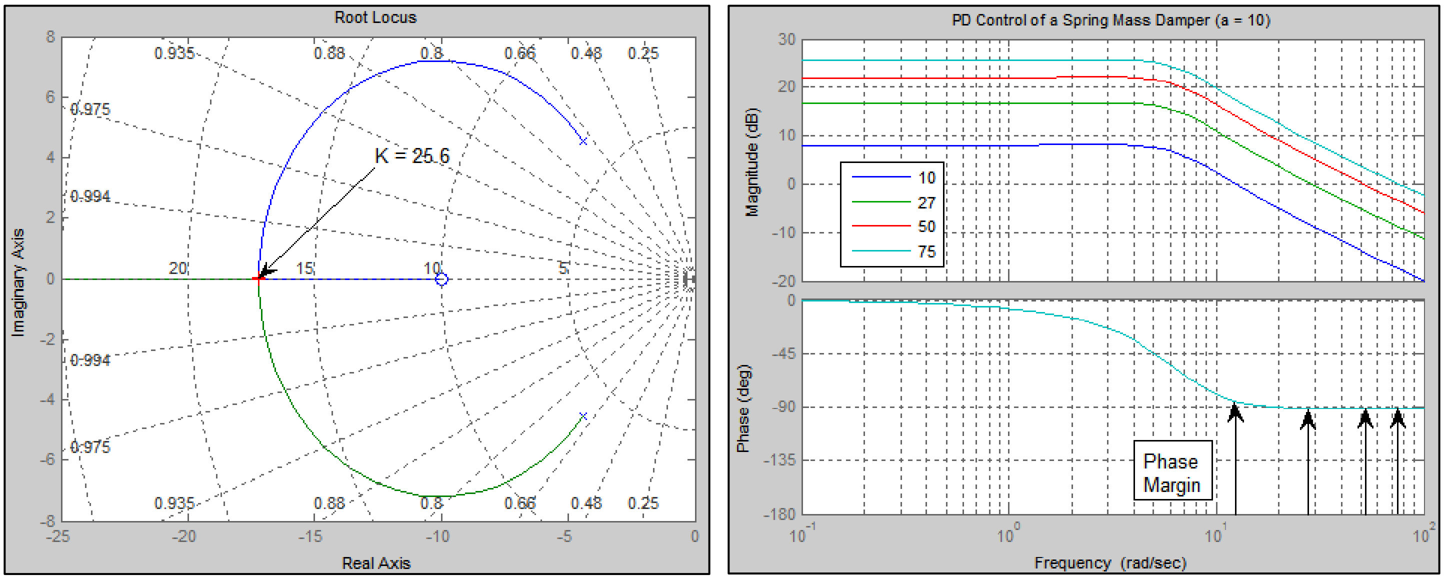



Fig. 6. Root Locus Diagram and Bode Diagram for ($GH(s)$) for PD Control $(a = 10)$

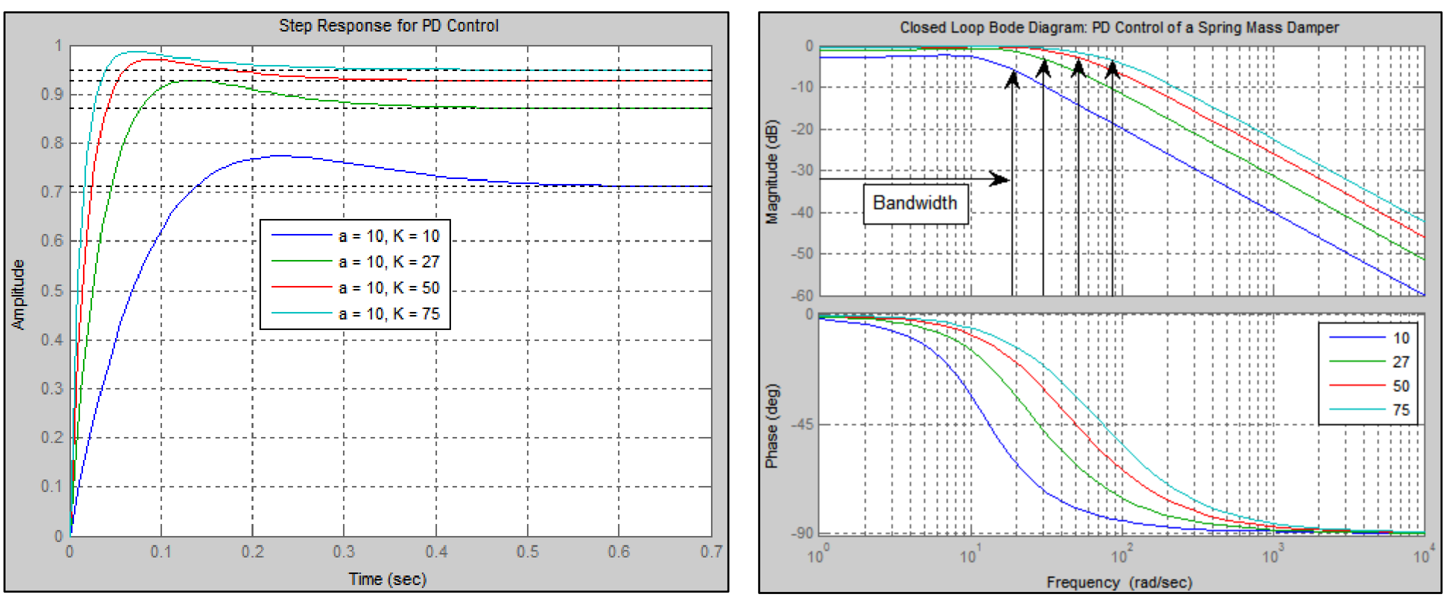



Fig. 7. Closed Loop Step Response and Bode Diagrams for PD Control $(a = 10)$

Proportional-Integral-Derivative Control

If ***proportional-integral-derivative (PID) control*** is used, then

$$\boxed{G_c(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_D(s^2 + as + b)}{s}} \tag{9}$$

The parameters $K_P$, $K_I$, and $K_D$ represent the ***proportional***, ***integral***, and ***derivative*** gains, the parameter $a = K_P / K_D$ is the ***ratio*** of the proportional and derivative gains, and the parameter $b = K_I / K_D$ is the ***ratio*** of the integral and derivative gains.

The loop and closed-loop transfer functions for this system are

$$\boxed{GH(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40)}} \qquad \boxed{\frac{X}{X_d}(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40) + K_D(s^2 + as + b)}} \tag{10}$$

Integral control makes the system ***type-one***, so it has ***zero*** steady-state error for a step input.

The root locus diagram for the closed-loop system (with $a = 15$ and $b = 50$) for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in ***Fig. 8***. The locations of the closed-loop poles for $K_D \approx 15.8$ are also shown. As the gain is ***increased***, the system becomes ***faster without significant losses*** in the ***phase margin***.

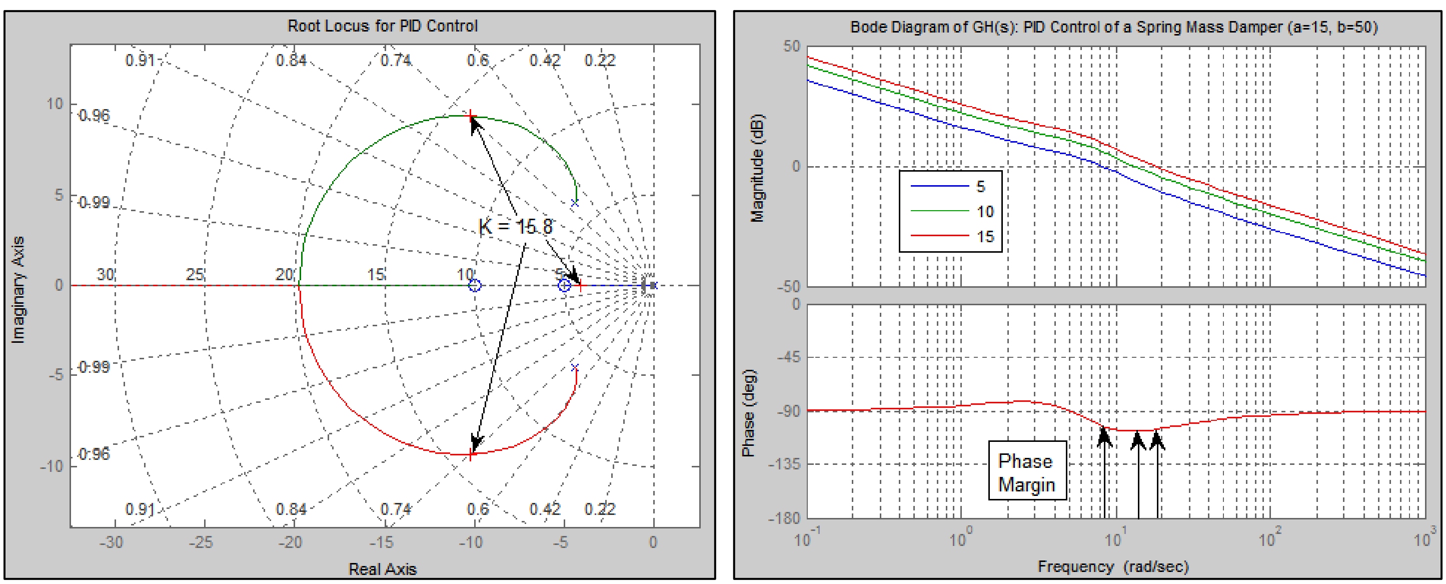



Fig. 8. Root Locus Diagram and Bode Diagram for ($GH(s)$) for PID Control $(a = 15, b = 50)$

***Fig. 9*** shows step responses and Bode diagrams of the closed-loop system for $a = 15$, $b = 50$, and derivative gains of $K_D = 5$, 10, and 15. Using both integral and derivative control has ***removed***

***steady-state error*** and ***decreased system settling times*** while maintaining a reasonable transient response.

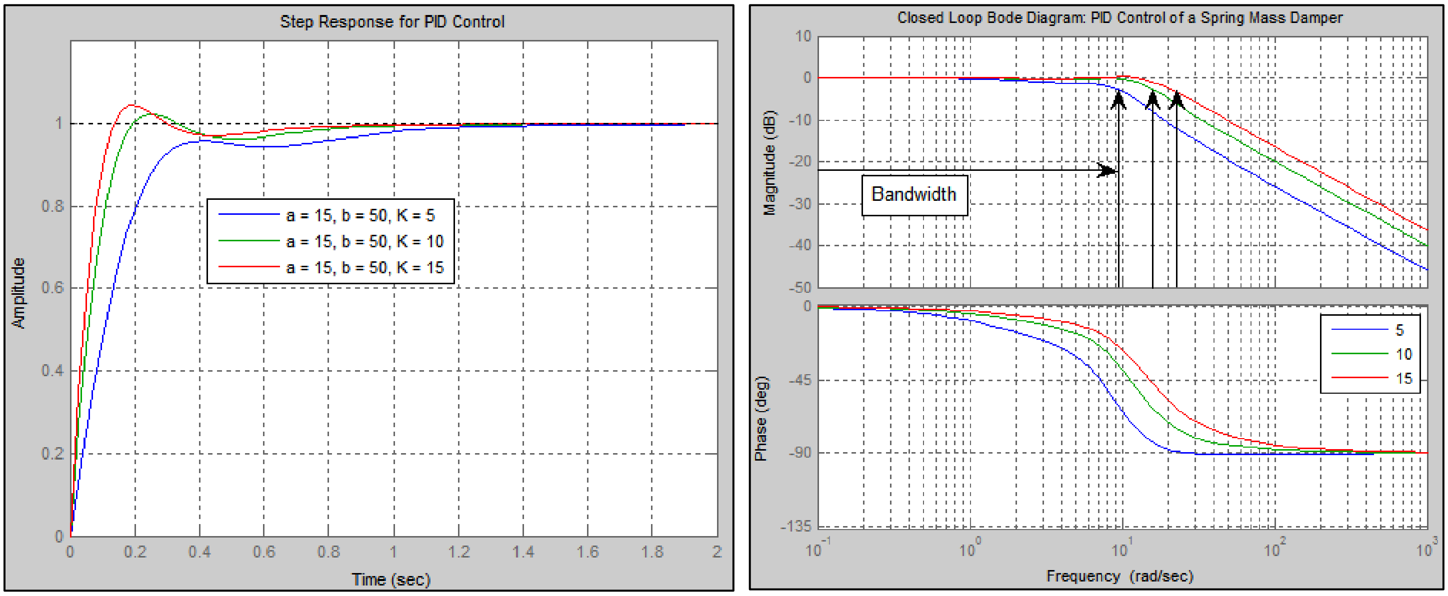



Fig. 9. Closed Loop Step Response and Bode Diagrams for PID Control $(a = 15, b = 50)$