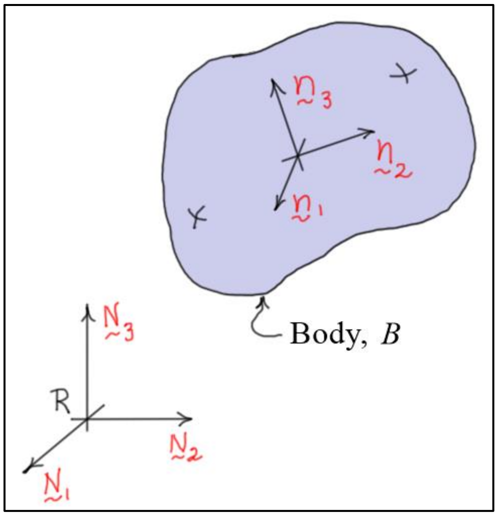
Intermediate Dynamics
Orientation Angles of a Rigid Body in Three Dimensions



To describe the ***general orientation*** of a rigid body in three dimensions, consider the rigid body shown in the figure. Here there are two reference frames – the ***base frame*** $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$, and the ***body-fixed frame*** $B:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3)$. In an arbitrary position, none of the unit vectors of the two frames are aligned. There are ***two methods*** for describing any orientation of B relative to the base frame R – orientation angles and orientation parameters.

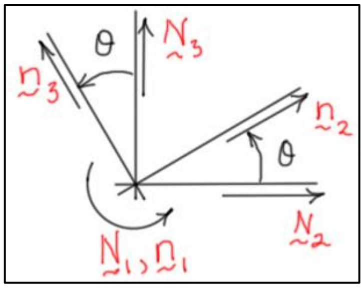
The first (and most commonly used) method of orienting a body in three dimensions involves the use of ***orientation angles***. These are easy to visualize, but they are ***not unique***, and they give rise to ***mathematical singularities*** in certain positions. The second method involves the use of ***Euler*** (or Euler-like) ***parameters***. These are ***not easy to visualize***; however, they are ***unique***, and they have ***no mathematical singularities***. The following notes discuss the use of ***orientation angles*** to describe angular position and motion of rigid bodies.

Simple Rotations

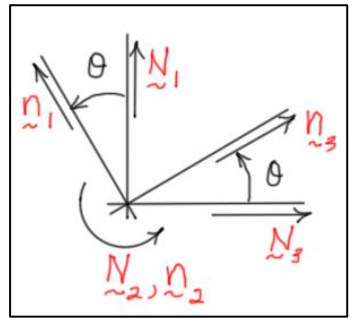
Simple rotations are defined as ***right-handed*** (or dextral) rotations about a single axis. For example, assume initially that the directions $(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3)$ are aligned with the directions $(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$. Then, an ***X-rotation*** is defined as a right-handed rotation of B about $\underset{\sim}{N}_1$ (or $\underset{\sim}{n}_1$), a ***Y-rotation*** as a right-handed rotation about $\underset{\sim}{N}_2$ (or $\underset{\sim}{n}_2$), and a ***Z-rotation*** as a right-handed rotation about $\underset{\sim}{N}_3$ (or $\underset{\sim}{n}_3$). For each of these simple rotations, the unit vectors of the two reference frames can be related to each other using the following matrix equations.

<u>*X*-rotation:</u>

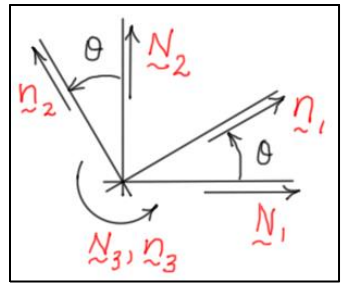
$$\begin{Bmatrix}\underset{\sim}{n}_1\\\underset{\sim}{n}_2\\\underset{\sim}{n}_3\end{Bmatrix}=\begin{bmatrix}1&0&0\\0&C_\theta&S_\theta\\0&-S_\theta&C_\theta\end{bmatrix}\begin{Bmatrix}\underset{\sim}{N}_1\\\underset{\sim}{N}_2\\\underset{\sim}{N}_3\end{Bmatrix}$$



Y-rotation: $\begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$



Z-Rotation: $\begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$

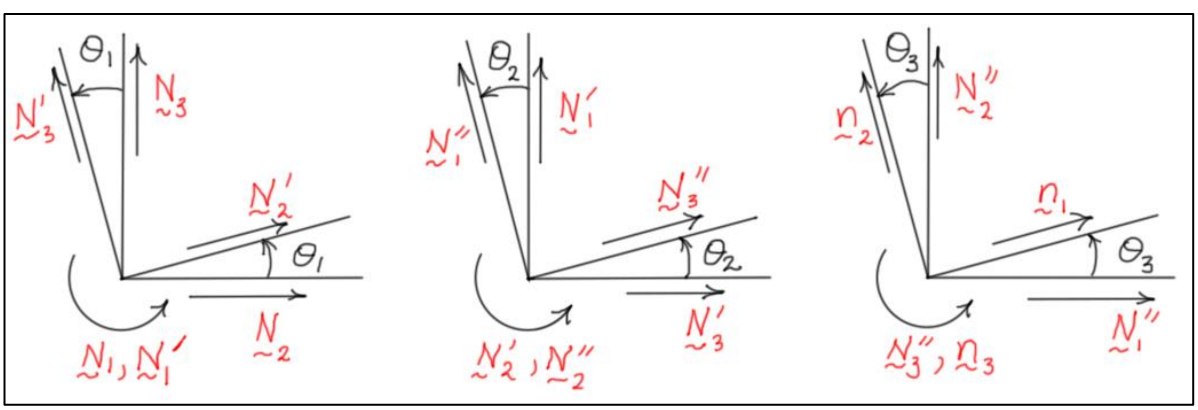


Here, S_θ and C_θ represent the ***sine*** and ***cosine*** of the rotation angle θ.

The coefficient matrices in the above equations are called "***transformation***" or "***rotation***" matrices. They are ***orthogonal*** matrices with a ***determinant*** of +1. As with all ***orthogonal matrices***, the ***inverses*** of these matrices are their ***transposes***. Hence, it is easy to ***invert*** the equations to express the base system unit vectors in terms of the body-fixed unit vectors.

General Orientations

A rigid body can be moved into ***any orientation*** (relative to the base frame) using a ***sequence*** of three simple rotations. These rotations can occur about the ***base-frame axes*** or the ***body-frame axes***. One common example is a body-fixed 1-2-3 rotation sequence. (Here, "1-2-3" has been used to stand for $\underset{\sim}{n}_1$, $\underset{\sim}{n}_2$, $\underset{\sim}{n}_3$ rotations.) To work through the sequence of rotations, ***intermediate reference frames*** are introduced as shown below.



The matrix equations for the three rotations are

$$\begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_1 & S_1 \\ 0 & -S_1 & C_1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = [R_1] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$$

$$\begin{Bmatrix} \underset{\sim}{N}_1'' \\ \underset{\sim}{N}_2'' \\ \underset{\sim}{N}_3'' \end{Bmatrix} = \begin{bmatrix} C_2 & 0 & -S_2 \\ 0 & 1 & 0 \\ S_2 & 0 & C_2 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix} = [R_2] \begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix}$$

$$\begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1'' \\ \underset{\sim}{N}_2'' \\ \underset{\sim}{N}_3'' \end{Bmatrix} = [R_3] \begin{Bmatrix} \underset{\sim}{N}_1'' \\ \underset{\sim}{N}_2'' \\ \underset{\sim}{N}_3'' \end{Bmatrix}$$

As before, S_i and C_i represent the ***sine*** and ***cosine*** of the rotation angle θ_i.

These equations can be ***combined*** to form a ***single matrix relationship*** between the base-fixed and the body-fixed unit vectors as follows.

$$\begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = [R_3][R_2][R_1] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = [R] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$$

So, for a body-fixed 1-2-3 rotation sequence, the ***transformation matrix*** that relates the unit vectors in the ***body-fixed frame*** to those in the ***base frame*** is

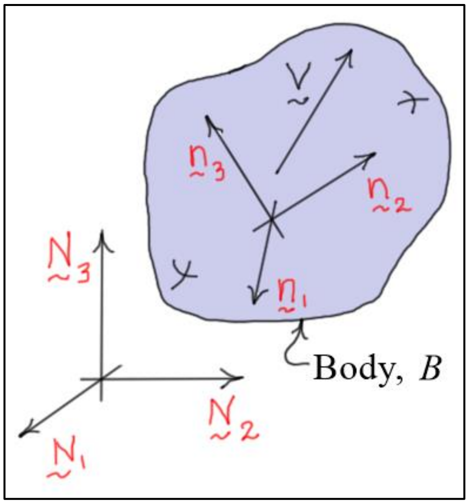
$$[R] = [R_3][R_2][R_1] = \begin{bmatrix} C_2C_3 & C_1S_3 + S_1S_2C_3 & S_1S_3 - C_1S_2C_3 \\ -C_2S_3 & C_1C_3 - S_1S_2S_3 & S_1C_3 + C_1S_2S_3 \\ S_2 & -S_1C_2 & C_1C_2 \end{bmatrix}$$

where the matrices $\left[R_i\right]$ $(i = 1,2,3)$ are defined in the above equations. Like the individual rotation matrices $\left[R_i\right]$, the matrix $[R]$ is an ***orthogonal matrix*** whose determinant is +1. So, again it is easy to ***invert*** the relationship between the unit vector sets.

Note: Transformation matrices for many different combinations of rotations are given in Appendix I of the text ***Spacecraft Dynamics*** by T. R. Kane, P. W. Likins, and D. A. Levinson, McGraw-Hill, 1983. In that text, the transformation matrix $[C]$ is the transpose of the matrix $[R]$ as defined above.

Relationship Between Vector Components

To find the relationship between ***vector components*** in the two different reference frames, consider the vector $\underset{\sim}{V}$ shown in the figure. If $\underset{\sim}{V}$ is ***most conveniently expressed*** in terms of unit vectors in the ***body frame*** $B:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3)$, then in matrix notation,



$$\underset{\sim}{V} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} [R] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$$

$$= \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$$

Comparing the matrices multiplying the base unit vectors gives

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} [R] = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$$

Right multiplying both sides by $[R]^T$ the transpose of $[R]$, and then taking the transpose of both sides gives

$$\boxed{\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = [R] \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix}}$$

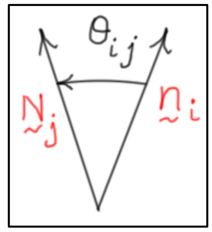
Pre-multiplying both sides of this equation with $[R]^T$ gives

$$\boxed{\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = [R]^T \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}}$$

Clearly, the vector components transform in the same way that the unit vectors do. Matrix $[R]$ transforms vector components in the ***base frame*** to those in the ***body frame***, and matrix $[R]^T$ transforms vector components in the ***body frame*** into the ***base frame***.

Transformation Matrices and Direction Cosines

The ***elements*** of a ***transformation matrix*** that relates the unit vectors of two different reference frames are the ***direction cosines*** of the various unit vector pairs. Given that r_{ij} $(i, j = 1,2,3)$ represent the ***elements*** of the transformation matrix $[R]$, it can be shown that



$$\boxed{r_{ij} = \underset{\sim}{n}_i \cdot \underset{\sim}{N}_j = C_{\theta_{ij}}}$$

where $C_{\theta_{ij}}$ $(i, j = 1,2,3)$ represent the ***cosines*** of the angles θ_{ij} $(i, j = 1,2,3)$ between the unit vectors $\underset{\sim}{n}_i$ $(i = 1,2,3)$ and $\underset{\sim}{N}_j$ $(j = 1,2,3)$.