

Intermediate Dynamics

Angular Velocity and Orientation Angles

When using *sequences of angles* to describe the *orientation* of a rigid body, the *summation rule* for *angular velocities* is used to find the angular velocity of the body. Consider the case where a 1-2-3 rotation sequence is used to describe the orientation of a body. In that case, the angular velocity of the body can be written as

$$\boxed{{}^R\omega_B = {}^R\omega_{R'} + {}^{R'}\omega_{R''} + {}^{R''}\omega_B = \dot{\theta}_1 \tilde{N}_1 + \dot{\theta}_2 \tilde{N}'_2 + \dot{\theta}_3 \tilde{N}''_3 = \dot{\theta}_1 \tilde{N}'_1 + \dot{\theta}_2 \tilde{N}''_2 + \dot{\theta}_3 \tilde{n}_3}$$

Recall that the “*primed*” axes are *intermediate* axes that are introduced so the angular velocity components are all “*simple*”. To make the form of ${}^R\omega_B$ most useful, it must be expressed in either the base reference frame ($R: \tilde{N}_1, \tilde{N}_2, \tilde{N}_3$) or in the body reference frame ($B: \tilde{n}_1, \tilde{n}_2, \tilde{n}_3$). For example, it is easy to show that the *body-fixed components* are

$$\boxed{\begin{aligned} \omega_1 &= \dot{\theta}_1 C_2 C_3 + \dot{\theta}_2 S_3 \\ \omega_2 &= -\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3 \\ \omega_3 &= \dot{\theta}_1 S_2 + \dot{\theta}_3 \end{aligned}} \quad \left({}^R\omega_B = \omega_1 \tilde{n}_1 + \omega_2 \tilde{n}_2 + \omega_3 \tilde{n}_3 \right)$$

These equations may be *inverted* to solve for $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in terms of ω_1 , ω_2 , and ω_3 to give

$$\boxed{\begin{aligned} \dot{\theta}_1 &= (\omega_1 C_3 - \omega_2 S_3) / C_2 \\ \dot{\theta}_2 &= \omega_1 S_3 + \omega_2 C_3 \\ \dot{\theta}_3 &= \omega_3 - S_2 (\omega_1 C_3 - \omega_2 S_3) / C_2 \end{aligned}}$$

Singularity Positions

The angular velocity equations shown above for the 1-2-3 orientation angle sequence are *singular* when $C_2 \triangleq \cos(\theta_2)$ is *zero*, that is, when the *second orientation angle* is 90 degrees. *All orientation angle sequences display such a singularity* at some position. This can cause problems for computer programs that use angle sequences to describe the orientation of rigid bodies.

Linearization of the 1-2-3 Orientation Angle Sequence about (0,0,0)

The above equations relating the *derivatives* of the *orientation angles* to the *angular velocity components* were derived assuming the body is in an arbitrary position. Now, it will be assumed that the body is exhibiting *small angular motions* about the orientation where *all angles* are *zero*, that is, around the equilibrium position $\theta_1 = \theta_2 = \theta_3 = \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = 0$. In this case, the angular velocity of the body can be described using equations *linearized* about the *equilibrium position*. The linearizations of the angular velocity expressions about the equilibrium position follow.

$$1. f(\theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2) = \dot{\theta}_1 C_2 C_3 + \dot{\theta}_2 S_3 \cong m_1 \theta_2 + m_2 \theta_3 + m_3 \dot{\theta}_1 + m_4 \dot{\theta}_2$$

$$m_1 = \left. \frac{\partial f}{\partial \theta_2} \right|_{eq} = (-\dot{\theta}_1 S_2 C_3)_{eq} = 0$$

$$m_2 = \left. \frac{\partial f}{\partial \theta_3} \right|_{eq} = (-\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3)_{eq} = 0$$

$$m_3 = \left. \frac{\partial f}{\partial \dot{\theta}_1} \right|_{eq} = (C_2 C_3)_{eq} = 1$$

$$m_4 = \left. \frac{\partial f}{\partial \dot{\theta}_2} \right|_{eq} = (S_3)_{eq} = 0$$

$$2. g(\theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2) = -\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3 \cong m_1 \theta_2 + m_2 \theta_3 + m_3 \dot{\theta}_1 + m_4 \dot{\theta}_2$$

$$m_1 = \left. \frac{\partial g}{\partial \theta_2} \right|_{eq} = (\dot{\theta}_1 S_2 S_3)_{eq} = 0$$

$$m_2 = \left. \frac{\partial g}{\partial \theta_3} \right|_{eq} = (-\dot{\theta}_1 C_2 C_3 - \dot{\theta}_2 S_3)_{eq} = 0$$

$$m_3 = \left. \frac{\partial g}{\partial \dot{\theta}_1} \right|_{eq} = (-C_2 S_3)_{eq} = 0$$

$$m_4 = \left. \frac{\partial g}{\partial \dot{\theta}_2} \right|_{eq} = (C_3)_{eq} = 1$$

$$2. h(\theta_2, \dot{\theta}_1, \dot{\theta}_3) = \dot{\theta}_1 S_2 + \dot{\theta}_3 \cong m_1 \theta_2 + m_2 \dot{\theta}_1 + m_3 \dot{\theta}_3$$

$$m_1 = \left. \frac{\partial h}{\partial \theta_2} \right|_{eq} = (\dot{\theta}_1 C_2)_{eq} = 0$$

$$m_2 = \left. \frac{\partial h}{\partial \dot{\theta}_1} \right|_{eq} = (S_2)_{eq} = 0$$

$$m_3 = \left. \frac{\partial h}{\partial \dot{\theta}_3} \right|_{eq} = 1$$

So, the *final result* of the *linearization* is that $\omega_1 \approx \dot{\theta}_1$, $\omega_2 \approx \dot{\theta}_2$, and $\omega_3 \approx \dot{\theta}_3$.