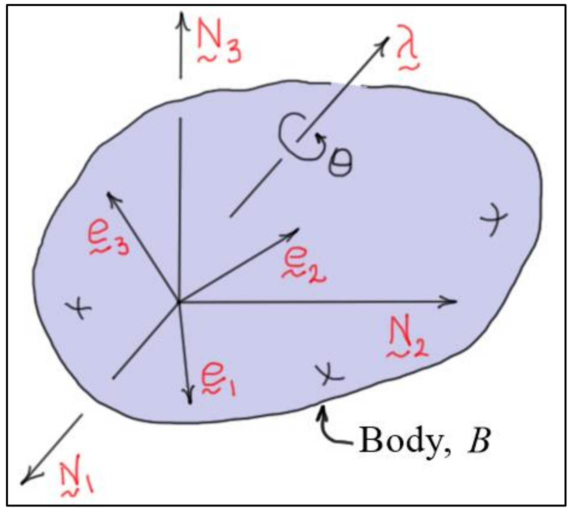
**Intermediate Dynamics**
**Orientation of a Rigid Body Using Euler Parameters**

## Euler's Theorem on Rotation

Consider the ***rigid body*** shown in the figure. Let $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$ represent the ***base reference frame*** and $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ represent the ***body-fixed reference frame*** and assume ***initially*** the two frames are ***aligned***. ***Euler's Theorem on Rotation*** states that body $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ can be moved into ***any arbitrary orientation*** relative to the base frame by a ***single rotation*** about ***some axis***.




In the diagram, $\theta$ represents the angle of rotation, and the ***unit vector*** $\underset{\sim}{\lambda}$ represents the direction (or axis) of rotation.

## Euler Parameters

The unit vector $\underset{\sim}{\lambda}$ and the angle $\theta$ can be related to a set of ***four parameters*** called the ***Euler parameters***. First, let $\underset{\sim}{\lambda}$ be expressed in terms of the base-frame unit vectors as

$$\boxed{\underset{\sim}{\lambda} = \lambda_1 \underset{\sim}{N}_1 + \lambda_2 \underset{\sim}{N}_2 + \lambda_3 \underset{\sim}{N}_3}$$

Then, the four Euler parameters are defined as follows.

$$\boxed{\begin{aligned} \varepsilon_1 &= \lambda_1 \sin(\theta/2) \\ \varepsilon_2 &= \lambda_2 \sin(\theta/2) \\ \varepsilon_3 &= \lambda_3 \sin(\theta/2) \\ \varepsilon_4 &= \cos(\theta/2) \end{aligned}} \qquad \text{(four Euler parameters)}$$

## Notes

1. The Euler parameters are ***not independent***, because $\boxed{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1}$.

2. It can be shown that the ***unit vectors*** in the two reference frames can be related as follows.

$$\begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = [R] \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix}$$

with

$$[R] = \begin{bmatrix} (\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) & 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) & (-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) & 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) & (-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) \end{bmatrix}$$

3. If the ***angular velocity*** of the body is resolved into ***components*** in the ***base reference frame***, that is, ${}^R\underset{\sim}{\omega}_B = \omega_1\underset{\sim}{N}_1 + \omega_2\underset{\sim}{N}_2 + \omega_3\underset{\sim}{N}_3$, then it can also be shown that

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{Bmatrix} = 2[E] \begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix} \qquad \text{or} \qquad \begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix} = \tfrac{1}{2}[E]^T \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{Bmatrix}$$

with

$$[E] = \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & -\varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & -\varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & -\varepsilon_3 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{bmatrix} \qquad \text{and} \qquad [E]^{-1} = [E]^T$$

***Similar expressions*** are true for the ***angular velocity components*** about the ***body-fixed axes***. Note that $[E]$ is an ***orthogonal matrix***.

4. Note that ***no singularities*** exist in the kinematic equations shown above, so many computer programs use Euler parameters (or Euler-like parameters) to ***avoid computational singularities***. They may, however, ***communicate*** with the analyst using orientation angles which are easier to visualize.