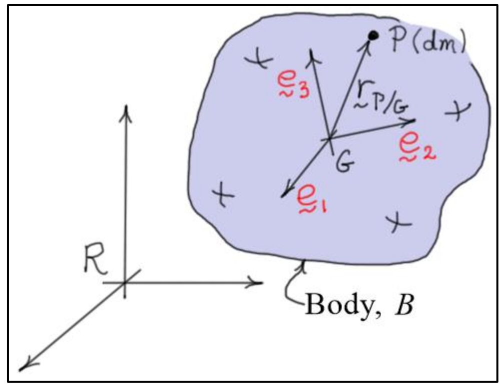
**Intermediate Dynamics**
**Moments and Products of Inertia and the Inertia Matrix**

## Moments of Inertia

A rigid body $B$ is shown in the diagram below. The unit vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ are fixed in the body and are directed along a ***convenient*** set of axes $(x, y, z)$ that pass through the mass center $G$. The ***moments of inertia*** of the body about these axes are defined as follows




$$I_{xx}^G = \int_B (y^2 + z^2)\, dm \qquad I_{yy}^G = \int_B (x^2 + z^2)\, dm \qquad I_{zz}^G = \int_B (x^2 + y^2)\, dm$$

Here, $x$, $y$, and $z$ are defined as the $\underset{\sim}{e}_i$ $(i = 1,2,3)$ components of $\underset{\sim}{r}_{P/G}$ the position vector of $P$ with respect to $G$, that is, $\underset{\sim}{r}_{P/G} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + z\underset{\sim}{e}_3$.

Moments of inertia of a body about an axis ***measure the distribution*** of the ***body's mass about that axis***. The smaller the inertia the more the mass is concentrated about the axis. Inertia values can be found either by ***measurement*** or by ***calculation***. Calculations are based on ***direct integration*** and/or on the "***body build-up***" technique. In the body build-up technique, ***inertias of simple shapes are added*** to estimate the inertia of a composite shape. The inertias of simple shapes (about their individual mass centers) are found in ***standard inertia tables***. These values are transferred to axes through the composite mass center using the ***Parallel Axes Theorem for Moments of Inertia***.

## Parallel Axes Theorem for Moments of Inertia

The inertia $(I_i^A)$ of a body about an axis $(i)$ through any point $(A)$ is equal to the inertia $(I_i^G)$ of the body about a parallel axis through the mass center $G$ plus the mass $(m)$ times the distance $(d_i)$ between the two parallel axes squared.

$$I_{ii}^A = I_{ii}^G + m\, d_i^2 \quad (i = x, y, \text{ or } z)$$

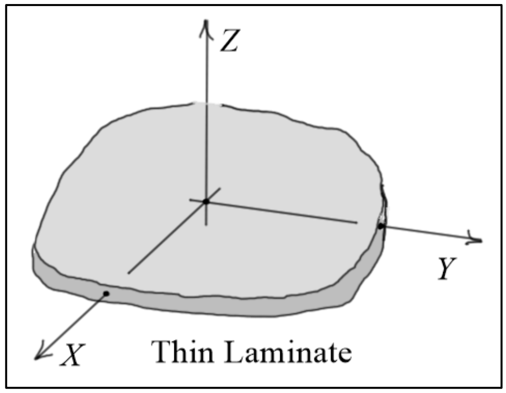
Note that moments of inertia are ***always positive***. From the parallel axes theorem, it is obvious that the ***minimum moments of inertia*** of a body occur about axes that pass through its ***mass center***.

**Products of Inertia**

The ***products of inertia*** of the rigid body are defined as

$$I_{xy}^{G} = \int_{B} (xy)\, dm \qquad I_{xz}^{G} = \int_{B} (xz)\, dm \qquad I_{yz}^{G} = \int_{B} (yz)\, dm$$

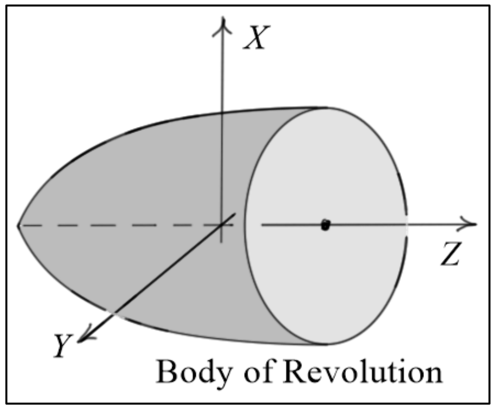
The products of inertia of a body are measures of ***symmetry***. ***If a plane is a plane of symmetry, then the products of inertia associated with any axis perpendicular to that plane are zero***. For example, consider the ***thin laminate*** shown. The middle plane of the laminate lies in the *XY*-plane so that half its thickness is above the plane and half is below. Hence, the ***XY-plane*** is a ***plane of symmetry*** and



Thin Laminate

$$I_{xz} = I_{yz} = 0$$

***Bodies of revolution*** have ***two planes*** of ***symmetry***. For the configuration shown, the *XZ* and *YZ* planes are planes of symmetry. Hence, ***all products of inertia are zero*** about the *X*, *Y*, and *Z* axes.



Body of Revolution

Products of inertia are found either by ***measurement*** or by ***calculation***. Calculations are based on ***direct integration*** or on the "***body build-up***" technique. In the body build-up technique, ***products of inertia of simple shapes are added*** to estimate the products of inertia of a composite shape. The products of inertia of simple shapes (about their individual mass centers) are found in ***standard inertia tables***. These values are transferred to axes through the composite mass center using the ***Parallel Axes Theorem for Products of Inertia***.

**Parallel Axes Theorem for Products of Inertia**

The product of inertia $(I_{ij}^{A})$ of a body about a pair of axes $(i, j)$ passing through any point $(A)$ is equal to the product of inertia $(I_{ij}^{G})$ of the body about a set of parallel axes through the mass center $G$ plus the mass $(m)$ times the product of the coordinates $(c_i c_j)$ of $G$ relative to $A$ (or $A$ relative to $G$) measured along those axes.

$$\boxed{I_{ij}^{A} = I_{ij}^{G} + m\, c_i\, c_j \qquad (i = x, y, \text{ or } z \;\text{ and }\; j = x, y, \text{ or } z)}$$

Products of inertia may be ***positive***, ***negative***, or ***zero***.

## The Inertia Matrix

The inertias of a body about a set of axes (passing through some point) are often collected into a single ***inertia matrix***. For example, the inertia matrix of a body about a set of axes through its mass center $G$ is defined as

$$\boxed{\left[I_G\right] = \begin{bmatrix} I_{11}^G & I_{12}^G & I_{13}^G \\ I_{21}^G & I_{22}^G & I_{23}^G \\ I_{31}^G & I_{32}^G & I_{33}^G \end{bmatrix} = \begin{bmatrix} I_{xx}^G & -I_{xy}^G & -I_{xz}^G \\ -I_{xy}^G & I_{yy}^G & -I_{yz}^G \\ -I_{xz}^G & -I_{yz}^G & I_{zz}^G \end{bmatrix}}$$

There is a ***different inertia matrix for each set of axes passing through a given point***. There is ***one set of directions*** for each point that ***renders the inertia matrix diagonal***. These directions are called ***principal directions*** (or ***principal axes***) of the body for that point. In general, the ***principal axes*** are ***different for each point*** in a body. Finally, note that ***all inertia matrices are symmetric***.

## The Inertia Dyadic

The inertias of a body about a set of axes (passing through some point) may also be collected into a single ***inertia dyadic***. For example, the inertia dyadic of a body about a set of axes through its mass center $G$ is defined as

$$\boxed{\underset{\approx}{I}_G = \sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}^G\, \underset{\sim}{e}_i\, \underset{\sim}{e}_j}$$

where $I_{ij}^G$ $(i, j = 1, 2, 3)$ are the ***elements*** of the ***inertia matrix***, and the ***vector product*** $\underset{\sim}{e}_i \underset{\sim}{e}_j$ is called a ***dyad***.

The ***dot product*** of a dyad with a vector results in another vector. A dyad can be pre-dotted or post-dotted with a vector, and the results of the two operations are generally different. There are many ***properties*** that dyads satisfy. Three ***properties*** that are useful when using inertia dyadics are listed below.

1. $\underset{\sim}{c} \cdot \left( \underset{\sim}{a}\underset{\sim}{b} \right) = \left( \underset{\sim}{c} \cdot \underset{\sim}{a} \right) \underset{\sim}{b}$ and $\left( \underset{\sim}{a}\underset{\sim}{b} \right) \cdot \underset{\sim}{c} = \underset{\sim}{a} \left( \underset{\sim}{b} \cdot \underset{\sim}{c} \right) = \left( \underset{\sim}{b} \cdot \underset{\sim}{c} \right) \underset{\sim}{a}$ $\Rightarrow \boxed{\underset{\sim}{c} \cdot \left( \underset{\sim}{a}\underset{\sim}{b} \right) \neq \left( \underset{\sim}{a}\underset{\sim}{b} \right) \cdot \underset{\sim}{c}}$
2. $\left( \underset{\sim}{a}\underset{\sim}{b} + \underset{\sim}{c}\underset{\sim}{d} \right) \cdot \underset{\sim}{e} = \left( \underset{\sim}{b} \cdot \underset{\sim}{e} \right) \underset{\sim}{a} + \left( \underset{\sim}{d} \cdot \underset{\sim}{e} \right) \underset{\sim}{c}$
3. $\underset{\sim}{a}\underset{\sim}{b} \neq \underset{\sim}{b}\underset{\sim}{a}$

The third property follows easily from the first property.

As noted above, the ***shorthand notation*** for the ***inertia dyadic*** of a body about its mass center $G$ is $\underset{\approx}{I}_G$. This notation is particularly useful when defining the ***angular momentum*** of a body.