

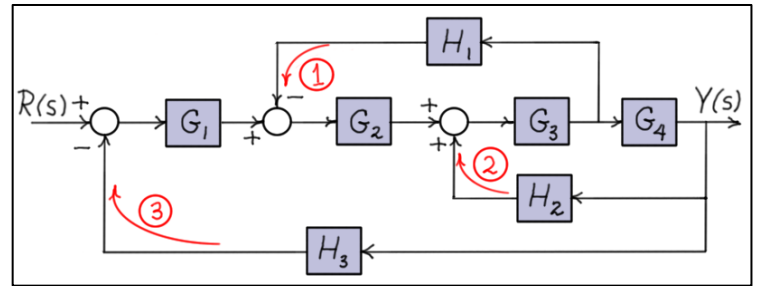
## Introductory Control Systems

### Block Diagram Reduction Examples

Example 1: R.C. Dorf and R.H. Bishop, *Modern Control Systems*, 11<sup>th</sup> Ed., Pearson Prentice-Hall, 2008.

Given: The system shown in the block diagram has **one input** signal,  $R(s)$ , and **one output** signal,  $Y(s)$ .

Find: Using **block diagram reduction**, find the transfer function  $\frac{Y}{R}(s)$ .



Example 2.7 from Dorf and Bishop

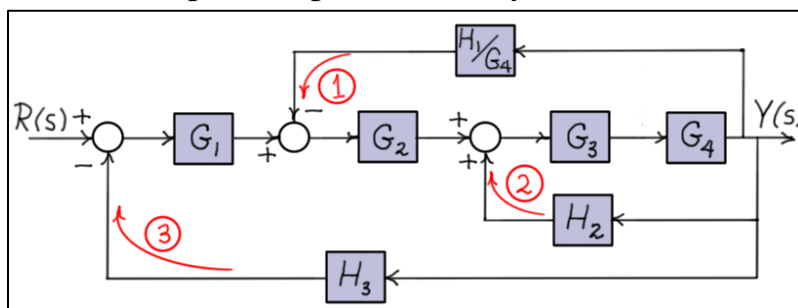
Solution:

First, it should be noted that there are **many solution paths** that can be taken to reduce complex systems using block diagram reduction techniques. The system shown here is not very complex, and yet, there are at least half a dozen approaches that can be taken. To encourage the reader to consider different solution paths, multiple approaches are presented here.

As shown, the system has **three feedback loops**. Loops 1 and 2 are **nested** in loop 3 but loops 1 and 2 are **overlapping**. To determine if two loops are overlapping, consider **collapsing** one of the two loops. If any signals required by the second loop would be **removed** by collapsing of the first, then the two loops are overlapping. In this system, if loop 1 is reduced, the summing block between  $G_2$  and  $G_3$  is eliminated, so the  $H_2$  feedback cannot be maintained. If loop 2 is reduced, the signal between  $G_3$  and  $G_4$  is eliminated, so the  $H_1$  feedback cannot be maintained.

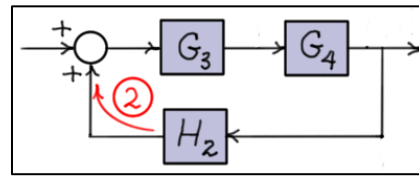
Approach 1:

In this first approach, loop 2 is **nested** within loop 1 by **expanding** loop 1 as shown in the diagram below. This is accomplished by moving the pick-off point of loop 1 to be behind  $G_4$ . This requires that feedback transfer function  $H_1$  be divided by  $G_4$ . Loop 2 is now nested within loop 1 which is nested within loop 3. Hence, reduction of loop 2, followed by reduction of loop 1, and finally the reduction of loop 3 will produce the system transfer function.



Loop 2 transfer function:

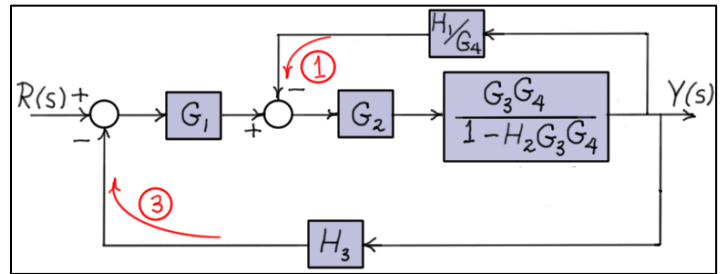
$$\frac{G}{1-GH} = \frac{G_3 G_4}{1-H_2 G_3 G_4} \quad (\text{positive feedback})$$



Transfer functions  $G_3$  and  $G_4$  are combined, then the positive feedback loop is collapsed. Now, substitute this result into the original diagram and then reduce the modified loop 1.

Loop 1 transfer function:

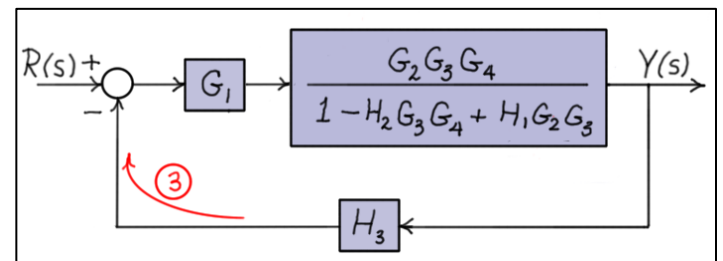
$$\frac{G}{1+GH} = \frac{\frac{G_2 G_3 G_4}{1-H_2 G_3 G_4}}{1 + \left( \frac{G_2 G_3 G_4}{1-H_2 G_3 G_4} \right) \left( \frac{H_1}{G_4} \right)} = \frac{G_2 G_3 G_4}{1-H_2 G_3 G_4 + H_1 G_2 G_3} \quad (\text{negative feedback})$$



Now, substitute this result into the diagram and reduce the modified loop 3 to find the system transfer function.

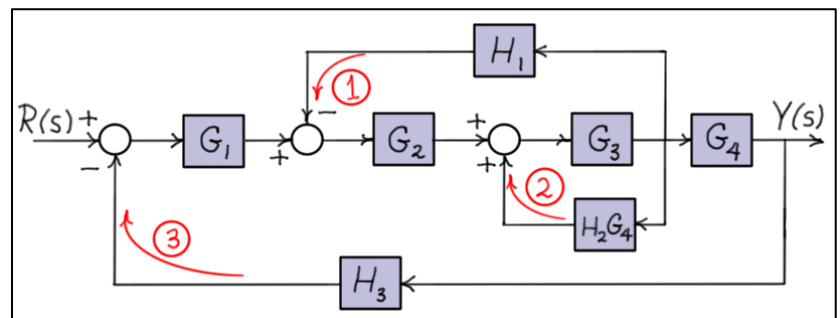
System transfer function:

$$\frac{Y}{R}(s) = \frac{G}{1+GH} = \frac{\frac{G_1 G_2 G_3 G_4}{1-H_2 G_3 G_4 + H_1 G_2 G_3}}{1 + \left( \frac{G_1 G_2 G_3 G_4}{1-H_2 G_3 G_4 + H_1 G_2 G_3} \right) H_3} = \frac{G_1 G_2 G_3 G_4}{1-H_2 G_3 G_4 + H_1 G_2 G_3 + H_3 G_1 G_2 G_3 G_4}$$



### Approach 2:

In this approach, loop 2 is nested within loop 1 by contracting loop 2 as shown in the diagram. This is accomplished by moving the pick-off point of loop 2 ahead of  $G_4$ . This requires that feedback transfer function  $H_2$  be multiplied by  $G_4$ .



As in the first approach, loop 2 is now *nested* within loop 1 which is *nested* within loop 3.

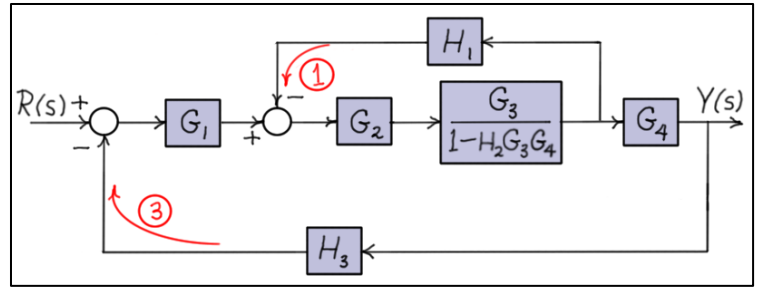
Loop 2 transfer function:

$$\frac{G}{1-GH} = \frac{G_3}{1-H_2G_3G_4} \quad (\text{positive feedback})$$

Now, substitute this result into the original diagram and then reduce the modified loop 1.

Loop 1 transfer function:

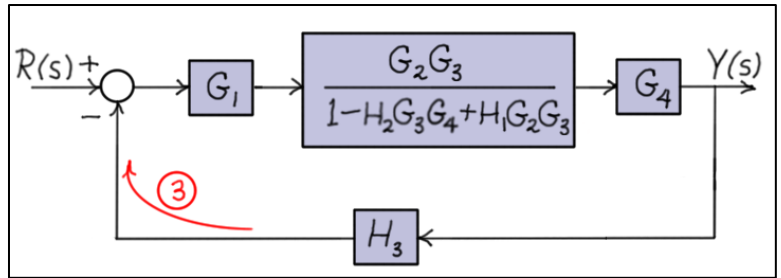
$$\frac{G}{1+GH} = \frac{\frac{G_2G_3}{1-H_2G_3G_4}}{1+H_1\left(\frac{G_2G_3}{1-H_2G_3G_4}\right)} = \frac{G_2G_3}{1-H_2G_3G_4+H_1G_2G_3} \quad (\text{negative feedback})$$



Substituting this result into the diagram and reducing loop 3 gives the system transfer function.

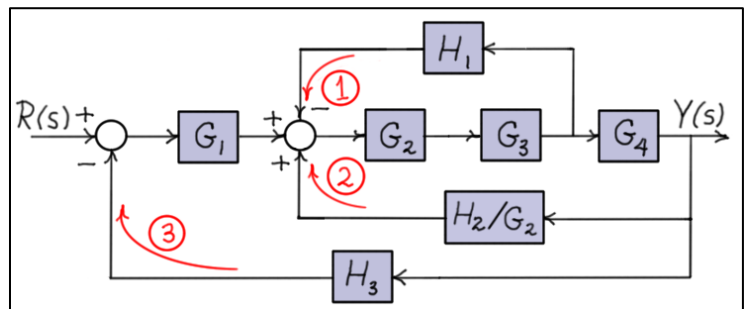
System transfer function:

$$\frac{Y}{R}(s) = \frac{G}{1+GH} = \frac{\frac{G_1G_2G_3G_4}{1-H_2G_3G_4+H_1G_2G_3}}{1+H_3\left(\frac{G_1G_2G_3G_4}{1-H_2G_3G_4+H_1G_2G_3}\right)} = \frac{G_1G_2G_3G_4}{1-H_2G_3G_4+H_1G_2G_3+H_3G_1G_2G_3G_4}$$



### Approach 3:

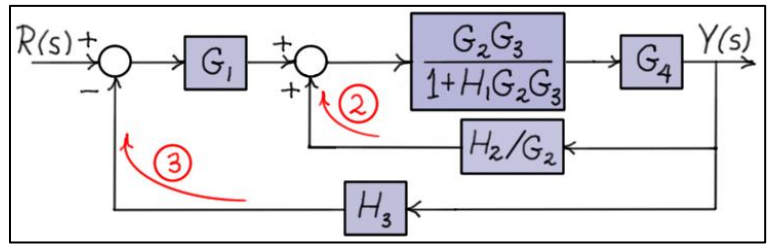
Loop 1 can also be *nested* within loop 2 by *expanding* loop 2 as shown in the diagram below. This is accomplished by moving the summing block of loop 2 ahead of  $G_2$ . This requires that feedback transfer function  $H_2$  be divided by  $G_2$ . Now, loop 1 is nested within loop 2 which is nested within loop 3.



Loop 1 transfer function:

$$\frac{G}{1+GH} = \frac{G_2G_3}{1+H_1G_2G_3} \quad (\text{negative feedback})$$

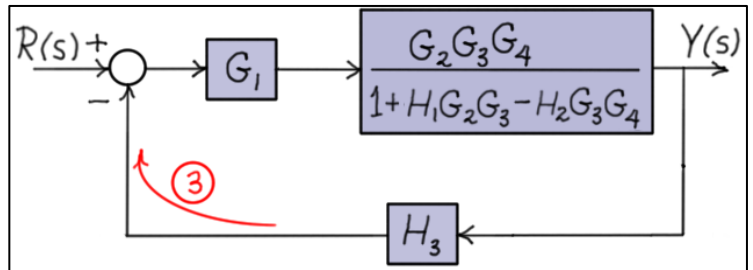
Now, substitute this result into the original diagram and then reduce the modified loop 2.



Loop 2 transfer function:

$$\frac{G}{1-GH} = \frac{\frac{G_2G_3G_4}{1+H_1G_2G_3}}{1-\frac{H_2}{G_2}\left(\frac{G_2G_3G_4}{1+H_1G_2G_3}\right)} = \frac{G_2G_3G_4}{1+H_1G_2G_3-H_2G_3G_4} \quad (\text{positive feedback})$$

Substituting this result into the diagram and reducing loop 3 gives the system transfer function.



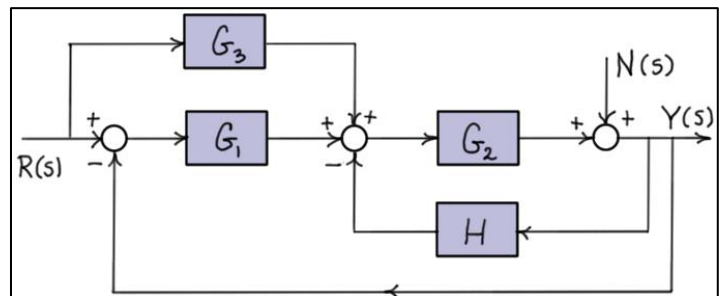
System transfer function:

$$\frac{Y}{R}(s) = \frac{G}{1+GH} = \frac{\frac{G_1G_2G_3G_4}{1+H_1G_2G_3-H_2G_3G_4}}{1+H_3\left(\frac{G_1G_2G_3G_4}{1+H_1G_2G_3-H_2G_3G_4}\right)} = \frac{G_1G_2G_3G_4}{1+H_1G_2G_3-H_2G_3G_4+H_3G_1G_2G_3G_4}$$

Example 2: Benjamin C. Kuo, *Automatic Control Systems*, 7<sup>th</sup> Ed., Prentice-Hall, 1995.

Given: The system shown in the block diagram has **two input** signals,  $R(s)$  and  $N(s)$ , and **one output** signal,  $Y(s)$ .

Find: Using **block diagram reduction**, find the transfer functions  $\frac{Y}{N}(s)$  and  $\frac{Y}{R}(s)$ .



Problem 3-7 from Kuo

Solution:

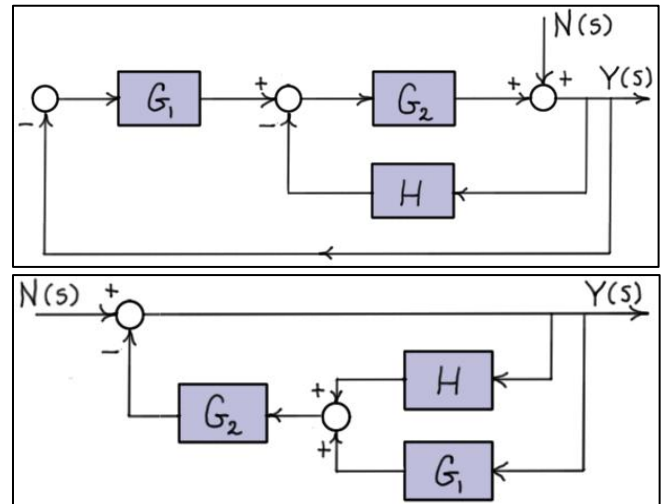
When a system has more than one input signal, transfer functions are found by **setting** all but one of the input signals to **zero**. So, in this case, if both input signals are non-zero, then the output can be written as

$$Y(s) = \left(\frac{Y}{N}(s)\right)N(s) + \left(\frac{Y}{R}(s)\right)R(s)$$

It is understood (but not always written) that  $R(s) \triangleq 0$  when finding  $\frac{Y}{N}(s)$ , and  $N(s) \triangleq 0$  when finding  $\frac{Y}{R}(s)$ . So, each **transfer function** contains information about a **single input, only**.

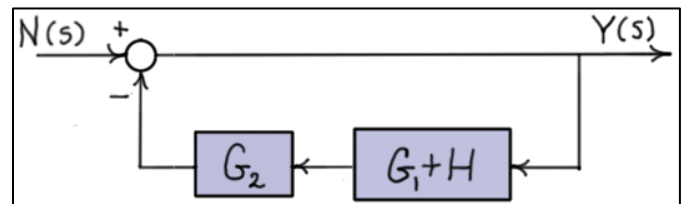
$\frac{Y}{N}(s)$ : Approach 1

Setting  $R(s) = 0$  in the diagram above gives the diagram on the right. The diagram shows the output  $Y(s)$  is fed back through the transfer functions  $G_1$  and  $H$ , and those two signals are combined at a summing block before passing through  $G_2$ . To show this more clearly, the system is **redrawn** with the input signal on the left and the output signal on the right.



Note that in the second diagram the signals are added **before** passing them through  $G_2$ , and then the combined signal is **negated** at the next summing block. This is **equivalent** to the sign changes made in the original diagram.

To simplify this diagram, note in the feedback loop that  $Y(s)$  is passed through both  $G_1$  and  $H$ , and then the two signals are added together. This process is the same as passing  $Y(s)$  through the transfer function  $G_1 + H$ .



The system transfer function can now be calculated as follows.

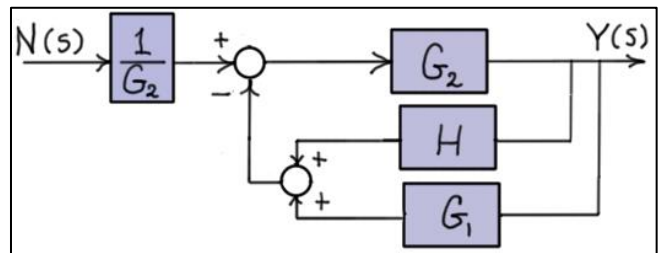
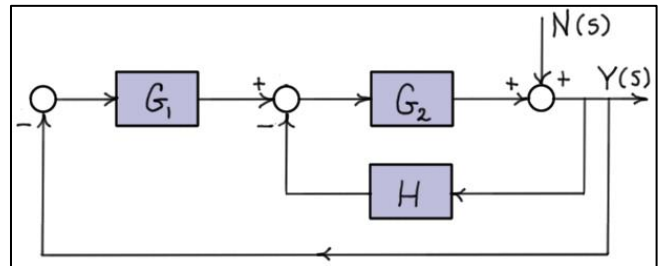
System transfer function:

$$\frac{Y}{N}(s) = \frac{G}{1+GH} = \frac{1}{1+G_2(G_1+H)} = \frac{1}{1+G_1G_2+HG_2}$$

Note here that the transfer functions  $G_1$  and  $G_2$  are **forward-path** transfer functions for the input  $R(s)$ , but they are **feedback-path** transfer functions for the input  $N(s)$ .

$$\frac{Y}{N}(s) : \text{Approach 2}$$

In the solution above,  $N(s)$  was kept at the same summing block as in the original diagram. As an alternative, consider moving  $N(s)$  to the middle summing block. To do this,  $N(s)$  must be divided by  $G_2$  as shown in the next diagram. Also, the signals for the  $H$  and  $G_1$  paths have been added together before negating the sum of the two signals at the next summing block. As noted above, the transfer functions in the **parallel paths** can be **summed**.



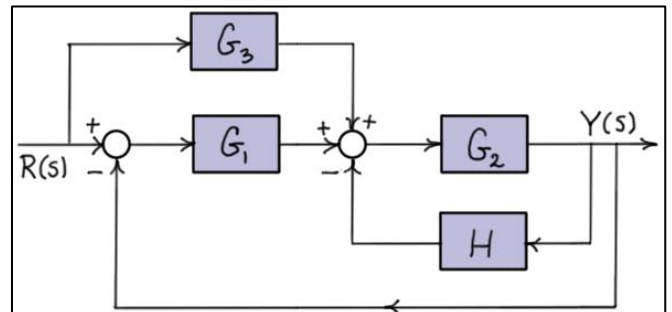
System transfer function:

$$\frac{Y}{N}(s) = \frac{1}{G_2} \left( \frac{G}{1+GH} \right) = \frac{1}{\cancel{G_2}} \left( \frac{\cancel{G_2}}{1+G_2(G_1+H)} \right) = \frac{1}{1+G_1G_2+HG_2} \quad (\text{as before})$$

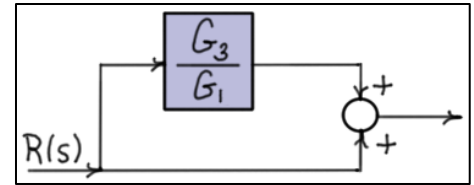
Note that  $N(s)$  could also be moved to the first summing block. In this case, it would have to be divided by the product  $G_1G_2$ .

$$\frac{Y}{R}(s) : \text{Approach 1}$$

Setting  $N(s) = 0$  in the original diagram gives the diagram on the right. To get started, the input signal  $R(s)$  needs to be **isolated** to a **single summing block**.

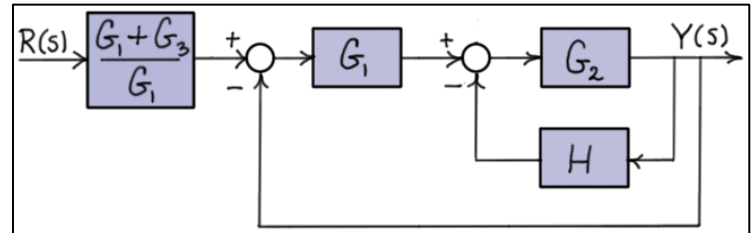


Moving the output of  $G_3$  to the summing block *ahead* of  $G_1$  requires the new signal be divided by  $G_1$ . Isolating the input diagram for  $R(s)$  gives the diagram on the right.



Input transfer function:  $1 + \frac{G_3}{G_1} = \frac{G_1 + G_3}{G_1}$  (parallel paths)

Substituting this result into the original diagram gives the diagram on the right. Now, the two feedback loops can be reduced, and that result can be cascaded with the input transfer function.



Inner feedback loop transfer function:  $\frac{G}{1+GH} = \frac{G_2}{1+HG_2}$  (negative feedback)

Outer feedback loop transfer function:

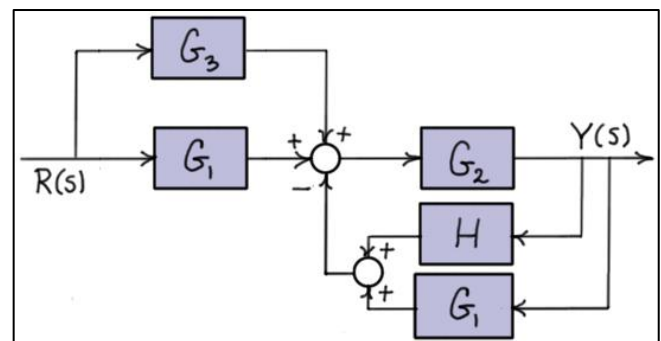
$$\frac{G}{1+GH} = \frac{G}{1+G} = \frac{\frac{G_1 G_2}{1+HG_2}}{1 + \left( \frac{G_1 G_2}{1+HG_2} \right)} = \frac{G_1 G_2}{1+HG_2 + G_1 G_2}$$
 (negative unity feedback)

System transfer function:

$$\frac{Y}{R}(s) = \left( \frac{G_1 + G_3}{G_1} \right) \left( \frac{\cancel{G_1} G_2}{1+HG_2 + G_1 G_2} \right) = \frac{G_2 (G_1 + G_3)}{1+HG_2 + G_1 G_2}$$

$\frac{Y}{R}(s)$  : Approach 2

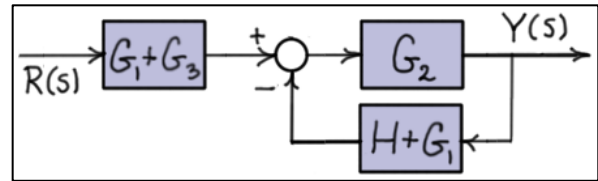
In this second approach, consider moving the outer feedback loop return to the second summing block as shown in the diagram. This requires the signal to be multiplied by  $G_1$ . In the diagram, the two feedback signals are added first and then negated at the next summing block.



Note here that the signal entering  $G_2$  (which ultimately gets fed back through the two feedback loops) is the same in this diagram as it is in the original. Hence, the diagrams are *equivalent*.



**Combining** the parallel paths into single transfer functions, the diagram can be redrawn as shown here. The system transfer function can now be calculated by reducing the closed loop and cascading the result with  $G_1 + G_3$ .

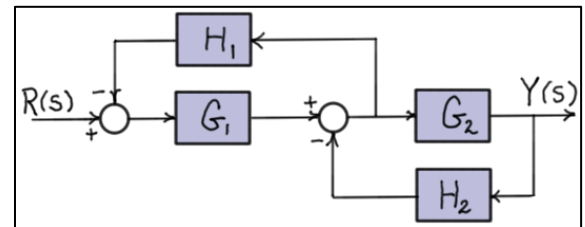


System transfer function: 
$$\frac{Y}{R}(s) = (G_1 + G_3) \left( \frac{G_2}{1 + G_2(H + G_1)} \right) = \frac{G_2(G_1 + G_3)}{1 + HG_2 + G_1G_2}$$
 (as before)

**Example 3:**

Given: The system shown in the block diagram has one input signal,  $R(s)$ , and one output signal,  $Y(s)$ .

Find: Using **block diagram reduction**, find the transfer function  $\frac{Y}{R}(s)$ .



Solution:

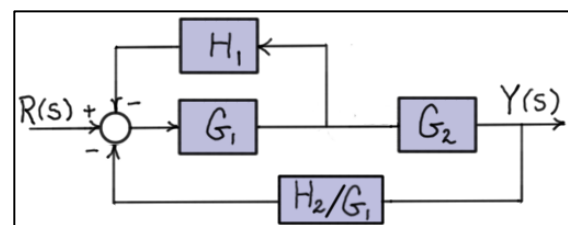
It is tempting with this system to simply **separate** the two feedback loops, but this approach leads to **incorrect results**. As the diagram indicates the two feedback loops are **overlapping**. If the **upper loop** is **collapsed**, the **second summing block** is **removed** and the **lower loop** cannot be **completed**, and if the **lower loop** is **collapsed** the **pick-off point** for the **upper loop** is **removed** and the **upper loop** cannot be **completed**.

When two feedback loops overlap, one of the loops should be expanded to **nest** the other loop within it. In this example, the pick-off point for  $H_1$  could be moved behind  $G_2$ , or the output of  $H_2$  could be moved ahead of  $G_1$ . The latter approach is taken here.

Inner loop transfer function:

$$\frac{G}{1 + GH} = \frac{G_1}{1 + G_1H_1} \quad (\text{negative feedback})$$

System transfer function:



$$\frac{Y}{R}(s) = \frac{G}{1 + GH} = \frac{\frac{G_1G_2}{1 + G_1H_1}}{1 + \left(\frac{H_2}{G_1}\right)\left(\frac{G_1G_2}{1 + G_1H_1}\right)} = \frac{G_1G_2}{1 + G_1H_1 + G_2H_2} \quad (\text{negative feedback})$$