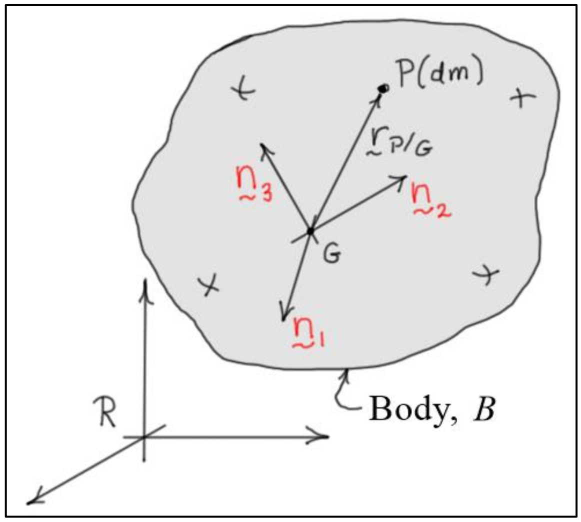
Intermediate Dynamics
Angular Momentum of a Rigid Body about its Mass Center

To calculate the ***angular momentum*** of a rigid body about its mass center, consider the body *B* shown in the figure. The point *P* represents ***any point*** in the body, dm represents the elemental mass of the body associated with *P*, and $\underset{\sim}{r}_{P/G}$ represents the position vector of *P* with respect to the mass-center *G*. The ***angular momentum*** of *B* about *G* is defined as follows



$$\boxed{\underset{\sim}{H}_G = \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P\right) dm}$$

Using the kinematic formula for ***two points fixed on a rigid body***, and the ***definition of center of mass*** (i.e. $\int_B \underset{\sim}{r}_{P/G}\, dm = 0$), this expression can be rewritten as follows

$$\begin{aligned}
\underset{\sim}{H}_G &= \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P\right) dm = \int_B \left(\underset{\sim}{r}_{P/G} \times \left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{v}_{P/G}\right)\right) dm \\
&= \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_G\right) dm + \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_{P/G}\right) dm \\
&= \left(\int_B \underset{\sim}{r}_{P/G}\, dm\right) \times {}^R\underset{\sim}{v}_G + \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_{P/G}\right) dm \\
&= \int_B \left(\underset{\sim}{r}_{P/G} \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right)\right) dm
\end{aligned}$$

Using the ***vector identity*** $\boxed{\underset{\sim}{a} \times \left(\underset{\sim}{b} \times \underset{\sim}{c}\right) = \left(\underset{\sim}{a} \cdot \underset{\sim}{c}\right)\underset{\sim}{b} - \left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right)\underset{\sim}{c}}$, the result above may be rewritten as

$$\underset{\sim}{H}_G = \int_B \left(\underset{\sim}{r}_{P/G} \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right)\right) dm = \int_B \left(\underset{\sim}{r}_{P/G} \cdot \underset{\sim}{r}_{P/G}\right) {}^R\underset{\sim}{\omega}_B\, dm - \int_B \left(\underset{\sim}{r}_{P/G} \cdot {}^R\underset{\sim}{\omega}_B\right) \underset{\sim}{r}_{P/G}\, dm$$

Setting $\boxed{\underset{\sim}{r}_{P/G} = x\,\underset{\sim}{n}_1 + y\,\underset{\sim}{n}_2 + z\,\underset{\sim}{n}_3}$ and $\boxed{{}^R\underset{\sim}{\omega}_B = \omega_1\,\underset{\sim}{n}_1 + \omega_2\,\underset{\sim}{n}_2 + \omega_3\,\underset{\sim}{n}_3}$, the above equation can be expanded as follows

$$\underset{\sim}{H}_G = \int_B \left(x^2 + y^2 + z^2\right)\left(\omega_1 \underset{\sim}{n}_1 + \omega_2 \underset{\sim}{n}_2 + \omega_3 \underset{\sim}{n}_3\right) dm - \int_B \left(x\omega_1 + y\omega_2 + z\omega_3\right)\left(x \underset{\sim}{n}_1 + y \underset{\sim}{n}_2 + z \underset{\sim}{n}_3\right) dm$$

$$= \int_B r^2 \left(\omega_1 \underset{\sim}{n}_1 + \omega_2 \underset{\sim}{n}_2 + \omega_3 \underset{\sim}{n}_3\right) dm - \int_B \left(x\omega_1 + y\omega_2 + z\omega_3\right)\left(x \underset{\sim}{n}_1 + y \underset{\sim}{n}_2 + z \underset{\sim}{n}_3\right) dm$$

$$= \int_B \left(r^2\omega_1 - x\left(x\omega_1 + y\omega_2 + z\omega_3\right)\right)\underset{\sim}{n}_1 dm + \int_B \left(r^2\omega_2 - y\left(x\omega_1 + y\omega_2 + z\omega_3\right)\right)\underset{\sim}{n}_2 dm + \int_B \left(r^2\omega_3 - z\left(x\omega_1 + y\omega_2 + z\omega_3\right)\right)\underset{\sim}{n}_3 dm$$

$$= \int_B \left(\left(y^2 + z^2\right)\omega_1 - xy\,\omega_2 - xz\,\omega_3\right)\underset{\sim}{n}_1 dm + \int_B \left(-xy\,\omega_1 + \left(x^2 + z^2\right)\omega_2 - yz\,\omega_3\right)\underset{\sim}{n}_2 dm + \int_B \left(-xz\,\omega_1 - yz\,\omega_2 + \left(x^2 + y^2\right)\omega_3\right)\underset{\sim}{n}_3 dm$$

Now, since the integrals do not depend on the angular velocity components (ω_1, ω_2, and ω_3), the above equation can be rewritten as

$$\underset{\sim}{H}_G = \left(\omega_1 \int_B \left(y^2 + z^2\right) dm + \omega_2 \int_B (-xy)\, dm + \omega_3 \int_B (-xz)\, dm\right)\underset{\sim}{n}_1 + \left(\omega_1 \int_B (-xy)\, dm + \omega_2 \int_B \left(x^2 + z^2\right) dm + \omega_3 \int_B (-yz)\, dm\right)\underset{\sim}{n}_2 + \left(\omega_1 \int_B (-xz)\, dm + \omega_2 \int_B (-yz)\, dm + \omega_3 \int_B \left(x^2 + y^2\right) dm\right)\underset{\sim}{n}_3$$

or

$$\boxed{\underset{\sim}{H}_G = \left(I_{xx}^G\omega_1 - I_{xy}^G\omega_2 - I_{xz}^G\omega_3\right)\underset{\sim}{n}_1 + \left(-I_{xy}^G\omega_1 + I_{yy}^G\omega_2 - I_{yz}^G\omega_3\right)\underset{\sim}{n}_2 + \left(-I_{xz}^G\omega_1 - I_{yz}^G\omega_2 + I_{zz}^G\omega_3\right)\underset{\sim}{n}_3}$$

Here, the integrals are now recognized as the ***moments*** and ***products of inertia*** of the body about axes parallel to $\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$ and passing through the mass center *G*.

Representation of Angular Momentum as a Matrix-Vector Product

The above result for the angular momentum vector $\underset{\sim}{H}_G$ is easier to remember when the following ***matrix-vector product*** is used.

$$\begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{n}_1 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{n}_2 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} I_{11}^G & I_{12}^G & I_{13}^G \\ I_{21}^G & I_{22}^G & I_{23}^G \\ I_{31}^G & I_{32}^G & I_{33}^G \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} I_{xx}^G & -I_{xy}^G & -I_{xz}^G \\ -I_{xy}^G & I_{yy}^G & -I_{yz}^G \\ -I_{xz}^G & -I_{yz}^G & I_{zz}^G \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

Representation of Angular Momentum as a Dyadic-Vector Product

The angular momentum vector $\underset{\sim}{H}_G$ can also be written as the "dot" product of the inertia dyadic with the angular velocity vector. That is,

$$\underset{\sim}{H}_G = \underset{\approx}{I}_G \cdot \underset{\sim}{\omega}$$

This is easily shown by substituting for $\underset{\approx}{I}_G$ and $\underset{\sim}{\omega}$ in this expression and expanding.

$$\underset{\sim}{H}_G = \left(\sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij}^G \underset{\sim}{n}_i \underset{\sim}{n}_j \right) \cdot \left(\sum_{k=1}^{3} \omega_k \underset{\sim}{n}_k \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \omega_k I_{ij}^G \underset{\sim}{n}_i \left(\underset{\sim}{n}_j \cdot \underset{\sim}{n}_k \right) =$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \omega_k I_{ij}^G \underset{\sim}{n}_i \delta_{jk} = \sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij}^G \omega_j \underset{\sim}{n}_i =$$

$$= \left(I_{11}^G \omega_1 + I_{12}^G \omega_2 + I_{13}^G \omega_3 \right) \underset{\sim}{n}_1 + \left(I_{21}^G \omega_1 + I_{22}^G \omega_2 + I_{23}^G \omega_3 \right) \underset{\sim}{n}_2 + \left(I_{31}^G \omega_1 + I_{32}^G \omega_2 + I_{33}^G \omega_3 \right) \underset{\sim}{n}_3$$

$$= \left(I_{xx}^G \omega_1 - I_{xy}^G \omega_2 - I_{xz}^G \omega_3 \right) \underset{\sim}{n}_1 + \left(-I_{xy}^G \omega_1 + I_{yy}^G \omega_2 - I_{yz}^G \omega_3 \right) \underset{\sim}{n}_2 + \left(-I_{xz}^G \omega_1 - I_{yz}^G \omega_2 + I_{zz}^G \omega_3 \right) \underset{\sim}{n}_3$$

In the above expression, δ_{jk} is the Kronecker delta symbol and is equal to ***one*** if $j = k$ and ***zero*** if $j \neq k$. This result shows that a ***dyadic-vector product*** can be represented using ***matrix multiplication***.

Question: When is $\underset{\sim}{H}_G$ parallel to $\underset{\sim}{\omega}$? In two dimensions? In three dimensions?