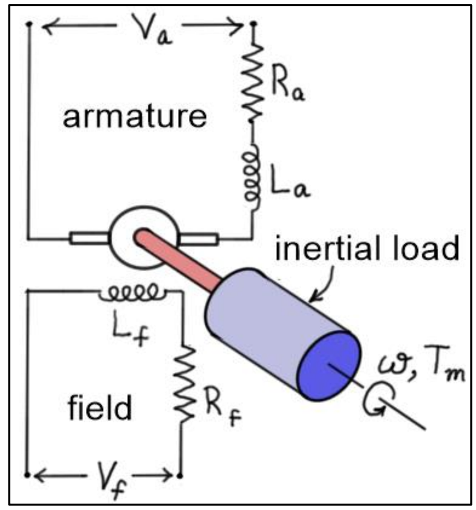
Introductory Control Systems
Armature Controlled DC Motor Transfer Functions

(Reference: Dorf and Bishop, Modern Control Systems, 9th Ed., Prentice-Hall, Inc. 2001)

- In an armature-current controlled DC motor, the field current i_f is held constant, and the armature current is controlled through the ***armature voltage*** V_a.
- The ***motor torque*** increases linearly with the armature current.

$$\boxed{T_m = K_{ma}\, i_a}$$

- K_{ma} is a ***constant*** that depends on a given motor. The ***transfer function*** from the input armature current to the resulting motor torque is



DC Motor

$$\boxed{\frac{T_m(s)}{I_a(s)} = K_{ma}} \qquad (1)$$

- The ***voltage/current relationship*** for the armature side of the motor is

$$\boxed{V_a = V_R + V_L + V_b = R_a i_a + L_a \left(\frac{di_a}{dt}\right) + V_b} \qquad (2)$$

- Here, V_b represents the "***back-EMF***" induced by the rotation of the armature windings in a magnetic field. V_b is proportional to the motor speed ω

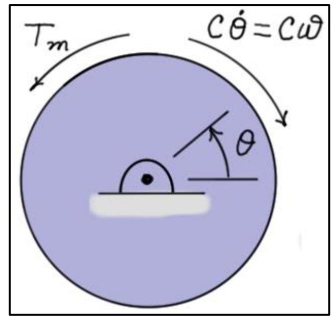
$$\boxed{V_b(s) = K_b \omega(s)}$$

K_b is the back-EMF coefficient.

- Taking Laplace transforms of Eq. (2) gives

$$\boxed{V_a(s) - V_b(s) = (R_a + L_a s) I_a(s)} \quad \text{or} \quad \boxed{V_a(s) - K_b \omega(s) = (R_a + L_a s) I_a(s)} \qquad (3)$$

- An equation describing the ***rotational motion*** of the ***inertial load*** is found by ***summing moments*** about the motor shaft.



Free Body Diagram of the Inertial Load

$$\sum M = T_m - c\omega = J\dot{\omega} \qquad \text{(CCW positive)}$$

or

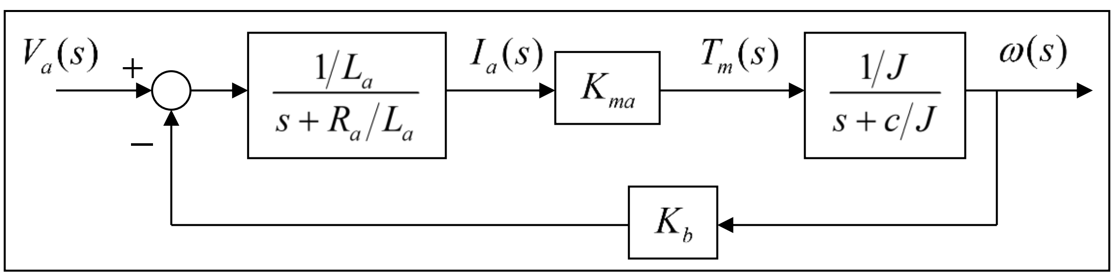
$$\boxed{J\dot{\omega} + c\omega = T_m} \qquad (4)$$

Eq. (4) is the differential equation of rotational motion, J is the ***moment of inertia*** of the load about the axis of rotation, and c is the ***damping coefficient***.

- The ***transfer function*** from the input ***motor torque*** to ***rotational speed*** changes is found by applying ***Laplace transforms*** to Eq. (4),

$$\boxed{\frac{\omega(s)}{T_m(s)} = \frac{(1/J)}{s + (c/J)}} \qquad \text{(1}^{\text{st}}\text{ order system)} \qquad (5)$$

- Equations (1), (3) and (5) together can be represented by the ***closed loop block diagram*** shown below.



- ***Block diagram reduction*** gives the transfer function from the input ***armature voltage*** to the resulting ***motor speed change***.

$$\boxed{\frac{\omega(s)}{V_a(s)} = \frac{(K_{ma}/L_a J)}{(s + R_a/L_a)(s + c/J) + (K_b K_{ma}/L_a J)}} \qquad \text{(2}^{\text{nd}}\text{ order system)} \qquad (6)$$

This is a second-order transfer function from armature voltage to motor speed changes.

- If the ***time constant*** of the ***electrical circuit*** is ***much smaller*** than the ***time constant*** of the ***inertial load dynamics***, the transfer function of Eq. (6) can be reduced to a ***first-order*** transfer function. Namely,

$$\boxed{\frac{\omega}{V_a}(s)=\frac{K_{ma}/R_aJ}{s+(cR_a+K_bK_{ma})/R_aJ}} \qquad \text{(1}^{\text{st}}\text{ order system)} \qquad (7)$$

- The ***transfer function*** from the input ***armature voltage*** to the resulting ***angular position change*** is found by multiplying Eqs. (6) and (7) by $\frac{1}{s}$.