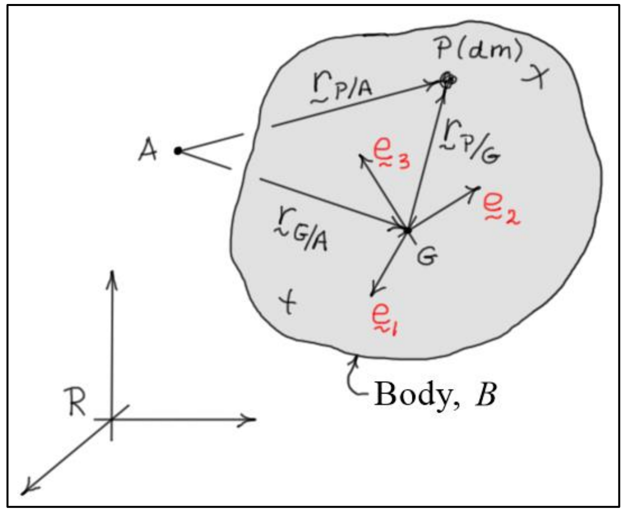
Intermediate Dynamics
Angular Momentum of a Rigid Body about an Arbitrary Point



The ***angular momentum*** of a rigid body B about its mass center G is defined as

$$\underset{\sim}{H}_G = \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P \right) dm \qquad (1)$$

In previous notes, it was shown that $\underset{\sim}{H}_G$ can be written in terms of the body's moments and products of inertia and the body-fixed angular velocity components as follows.

$$\underset{\sim}{H}_G = \left(I_{xx}^G \omega_1 - I_{xy}^G \omega_2 - I_{xz}^G \omega_3 \right) \underset{\sim}{n}_1 + \left(-I_{xy}^G \omega_1 + I_{yy}^G \omega_2 - I_{yz}^G \omega_3 \right) \underset{\sim}{n}_2 + \left(-I_{xz}^G \omega_1 - I_{yz}^G \omega_2 + I_{zz}^G \omega_3 \right) \underset{\sim}{n}_3$$

or

$$\underset{\sim}{H}_G = \underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B \qquad (2)$$

Here, $\underset{\approx}{I}_G$ represents the ***inertia dyadic*** (or matrix) of B about its mass-center G, and ${}^R\underset{\sim}{\omega}_B$ represents the ***angular velocity*** of the body.

The angular momentum of a rigid body about an ***arbitrary point*** A is similarly defined as

$$\underset{\sim}{H}_A = \int_B \left(\underset{\sim}{r}_{P/A} \times {}^R\underset{\sim}{v}_P \right) dm \qquad (3)$$

Substituting $\underset{\sim}{r}_{P/A} = \underset{\sim}{r}_{G/A} + \underset{\sim}{r}_{P/G}$ into the integrand, distributing the cross product over the sum, and expanding gives

$$\underset{\sim}{H}_A = \int_B \left(\left(\underset{\sim}{r}_{G/A} + \underset{\sim}{r}_{P/G} \right) \times {}^R\underset{\sim}{v}_P \right) dm = \int_B \left(\underset{\sim}{r}_{G/A} \times {}^R\underset{\sim}{v}_P \right) dm + \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P \right) dm$$

$$= \underset{\sim}{r}_{G/A} \times \left(\int_B {}^R\underset{\sim}{v}_P \, dm \right) + \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P \right) dm$$

$$= \underset{\sim}{r}_{G/A} \times \left(m \, {}^R\underset{\sim}{v}_G \right) \quad + \quad \underset{\sim}{H}_G$$

Or,

$$\underset{\sim}{H}_A = \underset{\sim}{H}_G \; + \; \underset{\sim}{r}_{G/A} \times m \, {}^R\underset{\sim}{v}_G \qquad (4)$$

So, the angular momentum of the body about an arbitrary point A is equal to the angular momentum of the body about its ***mass-center*** G plus the ***moment*** of the body's ***linear momentum*** about A.

Special Case: Motion about a Fixed-Point on the Body

If some point O of the rigid body is ***fixed*** so that the body ***pivots*** about this point, the velocity of the mass center may be written as ${}^R\underset{\sim}{v}_G = {}^R\underset{\sim}{v}_O + {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{G/O} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{G/O}$. Substituting this result into Eq. (4) above and combining terms, it can be shown that the angular momentum of the body about point O can be written as

$$\boxed{\underset{\sim}{H}_O = \underset{\approx}{I}_O \cdot {}^R\underset{\sim}{\omega}_B} \tag{5}$$

Here, $\underset{\approx}{I}_O$ represents the inertia dyadic (or matrix) about the fixed-point O.