Introductory Control Systems
Proportional Position Control of a Spring-Mass-Damper (SMD)

- Fig. 1 shows a ***spring-mass-damper*** system with a ***force actuator*** for ***position control***. The spring has stiffness k, the damper has coefficient b, the block has mass m, and the position of the mass is measured by the variable x.

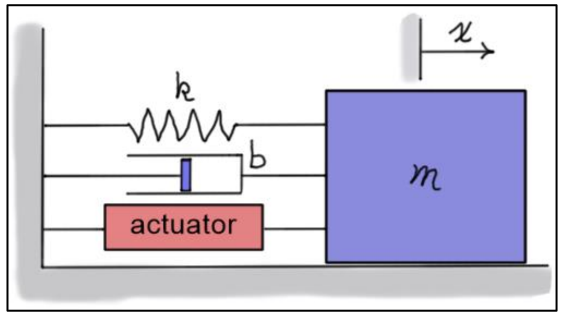
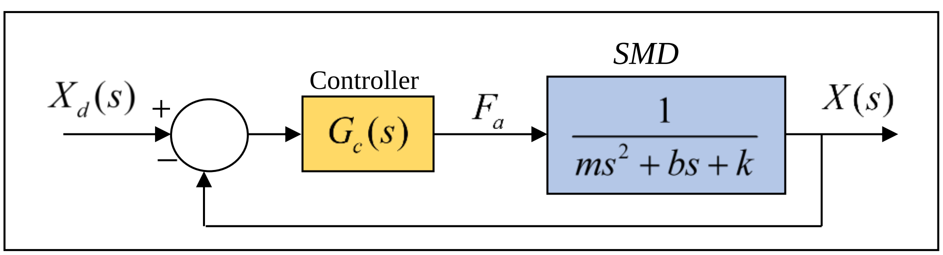


Figure 1. Spring-Mass-Damper System with Force Actuator

- The ***transfer function*** of the SMD: ***input*** is actuating force F_a and ***output*** is position x

$$\boxed{\frac{X}{F_a}(s) = \frac{1}{ms^2 + bs + k}} \qquad (1)$$

- This is a ***second-order*** transfer function that may be ***over-damped***, ***under-damped***, or ***critically damped*** depending on the values of the parameters m, b, and k.
- Assuming ***ideal actuator*** and ***sensor*** responses, the closed-loop ***position control*** of the SMD can be described using the following block diagram. Here, X_d represents the ***desired position***, X represents the ***actual position***, and $G_c(s)$ represents the transfer function of the ***controller***.



- If simple ***proportional control*** is used, then $G_c(s) = K$. Using ***block diagram reduction***, the transfer function for this case is found to be

$$\boxed{\frac{X}{X_d}(s) = \frac{K}{ms^2 + bs + (k+K)}} \qquad (2)$$

- The ***closed-loop*** system is also a ***second-order*** system, but it is ***not*** quite the same as the open-loop system. Using Eq. (2), the following observations can be made.

- **Observations**:
  1. The "stiffness" of the closed-loop system is $(k+K)>k$ for all positive gains K. However, the "mass" and "damping" coefficients are the ***same*** as for the open-loop SMD. This will give the closed-loop system a ***higher natural frequency*** ω_n.
  2. The ***damping ratio*** ζ of the closed-loop system is ***smaller*** than that of the open-loop system, because the product $2\zeta\omega_n = b/m$ is the ***same*** for both systems, and ω_n is ***higher*** for the closed-loop system.
  3. For a ***unit step command***, the final value of $x(t)$ is

$$\boxed{x_{ss} = \lim_{s\to 0}\left(\cancel{s} \cdot \frac{1}{\cancel{s}} \cdot \frac{K}{ms^2 + bs + (K+k)} \right) = \frac{K}{K+k} < 1}$$

- The general conclusion here is that proportional control can be used to alter the system's response. The question is whether the altered response is an acceptable response. To examine this question, consider the following example.

Example: $\boxed{m = 1 \text{ slug}}$, $\boxed{b = 8.8 \text{ (lb-s/ft)}}$, and $\boxed{k = 40 \text{ (lb/ft)}}$

- Using these values, the ***natural frequency*** and ***damping ratio*** of the open-loop system are $\boxed{\omega_n = \sqrt{40} = 6.325 \text{ (rad/s)} \approx 1 \text{ (Hz)}}$ and $\boxed{\zeta = \frac{8.8}{2\sqrt{40}} = 0.696 \approx 0.7}$.

- The following table lists the ***natural frequencies***, ***damping ratios***, and ***final values*** of the closed-loop system to a ***unit step input*** for various values of controller parameter K.

Gain, K	Natural Frequency ω_n (rad/s), (Hz)	Damping Ratio ζ	Final Value to a Unit Step
100	11.83, 1.88	0.37	0.71
500	23.24, 3.7	0.19	0.93
1000	32.25, 5.13	0.14	0.96
2000	45.17, 7.19	0.1	0.98

- Note that the values shown in the table ***corroborate*** the observations listed above. Unfortunately, for this system, when the gain K is high enough to produce a final value close to one, the damping ratio is quite small. So, ***proportional control*** is not necessarily a good choice for this system. As will be seen later in these notes, the addition of ***integral*** and ***derivative*** terms to the controller make for a better closed-loop response.

- Fig. 2 below shows the ***closed loop step response*** of the closed-loop system for gains K of 100, 500, and 2000. Note, as expected, that as K ***increases***, the ***frequency of response*** and ***final value*** both ***increase***, and the ***damping ratio decreases***.

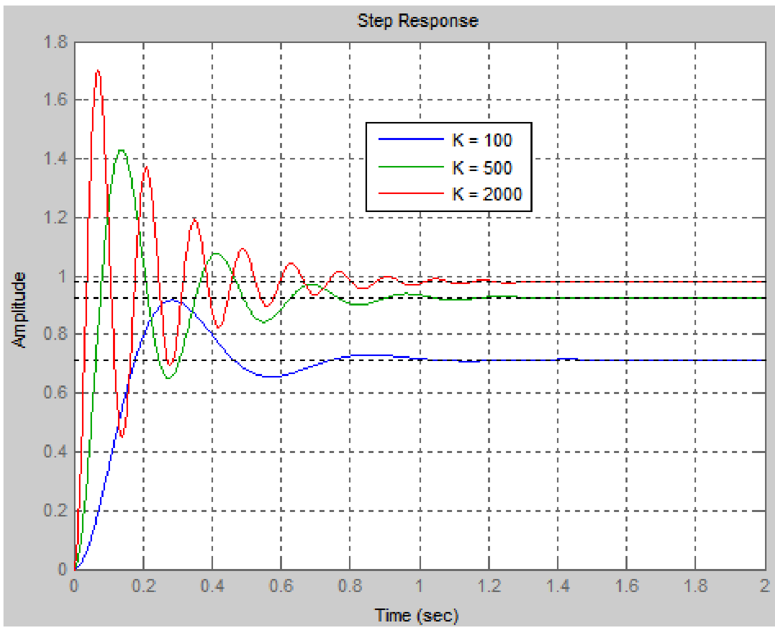


Figure 2. Step Response of the Closed-Loop System for Various Gains