

Intermediate Dynamics

Kinetic Energy of a Rigid Body

The figure depicts a rigid body B moving relative to a fixed frame R . The *kinetic energy* of B is defined as follows:

$$K = \int_B \frac{1}{2} \left({}^R \underline{v}_P \cdot {}^R \underline{v}_P \right) dm$$

A more *useful definition* can be derived by relating the velocity of P to the velocity of the mass center G . Using the relative velocity equation, the integrand can be rewritten as

$$\begin{aligned} {}^R \underline{v}_P \cdot {}^R \underline{v}_P &= \left({}^R \underline{v}_P \right)^2 = \left({}^R \underline{v}_G + {}^R \underline{v}_{P/G} \right)^2 = \left({}^R \underline{v}_G + \left({}^R \underline{\omega}_B \times {}^R \underline{r}_{P/G} \right) \right)^2 \\ &= \left({}^R \underline{v}_G \right)^2 + 2 {}^R \underline{v}_G \cdot \left({}^R \underline{\omega}_B \times {}^R \underline{r}_{P/G} \right) + \left({}^R \underline{\omega}_B \times {}^R \underline{r}_{P/G} \right)^2 \end{aligned}$$

Substituting this result into the integral gives the following three integrals:

$$1. \int_B \frac{1}{2} \left({}^R \underline{v}_G \right)^2 dm = \frac{1}{2} \left({}^R \underline{v}_G \right)^2 \int_B dm = \frac{1}{2} m \left({}^R \underline{v}_G \right)^2$$

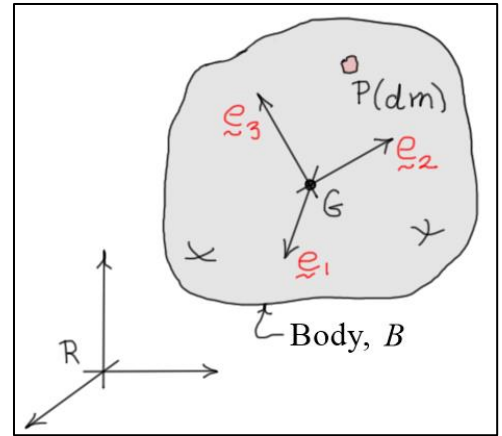
$$2. \int_B 2 {}^R \underline{v}_G \cdot \left({}^R \underline{\omega}_B \times {}^R \underline{r}_{P/G} \right) dm = 2 {}^R \underline{v}_G \cdot \left({}^R \underline{\omega}_B \times \left(\int_B {}^R \underline{r}_{P/G} dm \right) \right) = 0 \quad (\text{definition of mass center})$$

3. Setting ${}^R \underline{r}_{P/G} = x \underline{e}_1 + y \underline{e}_2 + z \underline{e}_3$ and ${}^R \underline{\omega}_B = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3$, the integrand of the third integral can be expanded as follows:

$$\begin{aligned} \left({}^R \underline{\omega}_B \times {}^R \underline{r}_{P/G} \right)^2 &= \left(\omega_2 z - \omega_3 y \right)^2 + \left(\omega_3 x - \omega_1 z \right)^2 + \left(\omega_1 y - \omega_2 x \right)^2 \\ &= \omega_1^2 \left(y^2 + z^2 \right) + \omega_2^2 \left(x^2 + z^2 \right) + \omega_3^2 \left(x^2 + y^2 \right) \\ &\quad - 2\omega_1 \omega_2 xy - 2\omega_1 \omega_3 xz - 2\omega_2 \omega_3 yz \end{aligned}$$

Substituting into the integral gives

$$\begin{aligned} \int_B \frac{1}{2} \left({}^R \underline{\omega}_B \times {}^R \underline{r}_{P/G} \right)^2 dm &= \frac{1}{2} \omega_1^2 \int_B \left(y^2 + z^2 \right) dm + \frac{1}{2} \omega_2^2 \int_B \left(x^2 + z^2 \right) dm + \frac{1}{2} \omega_3^2 \int_B \left(x^2 + y^2 \right) dm \\ &\quad - \omega_1 \omega_2 \int_B xy dm - \omega_1 \omega_3 \int_B xz dm - \omega_2 \omega_3 \int_B yz dm \\ &= \frac{1}{2} \omega_1^2 I_{xx}^G + \frac{1}{2} \omega_2^2 I_{yy}^G + \frac{1}{2} \omega_3^2 I_{zz}^G - \omega_1 \omega_2 I_{xy}^G - \omega_1 \omega_3 I_{xz}^G - \omega_2 \omega_3 I_{yz}^G \\ &= \frac{1}{2} {}^R \underline{\omega}_B \cdot \underline{H}_G \end{aligned}$$



Adding the three results gives

$$\boxed{K = \frac{1}{2}m\left({}^R\mathcal{V}_G\right)^2 + \frac{1}{2}{}^R\boldsymbol{\omega}_B \cdot \mathbf{H}_G = \frac{1}{2}m\left({}^R\mathcal{V}_G\right)^2 + \frac{1}{2}{}^R\boldsymbol{\omega}_B \cdot \mathbf{I}_G \cdot {}^R\boldsymbol{\omega}_B} \quad (\text{general motion})$$

Special Case: Motion about a Fixed Point in the Body

If there is a point O within the body that is fixed so that the body pivots about O , then the above result can be simplified as follows.

$$\begin{aligned} \left({}^R\mathcal{V}_G\right)^2 &= \left({}^R\boldsymbol{\omega}_B \times \mathbf{r}_{G/O}\right)^2 = \left(y_G^2 + z_G^2\right)\omega_1^2 + \left(x_G^2 + z_G^2\right)\omega_2^2 + \left(x_G^2 + y_G^2\right)\omega_3^2 \\ &\quad - 2\omega_1\omega_2x_Gy_G - 2\omega_1\omega_3x_Gz_G - 2\omega_2\omega_3y_Gz_G \end{aligned}$$

Substituting this result into the boxed equation above and combining terms, it can be shown that the kinetic energy can be reduced to the form

$$\boxed{K = \frac{1}{2}{}^R\boldsymbol{\omega}_B \cdot \mathbf{H}_O = \frac{1}{2}{}^R\boldsymbol{\omega}_B \cdot \mathbf{I}_O \cdot {}^R\boldsymbol{\omega}_B} \quad (\text{motion about a fixed point in the body})$$

Here, \mathbf{I}_O is the inertia dyadic (or matrix) for a set of axes through the fixed-point O .