Intermediate Dynamics
Angular Momentum and Kinetic Energy of a Simple Crank Shaft

Intermediate Dynamics – Example #11

The figure shows a ***simple crank shaft*** consisting of ***seven segments***, each considered to be a ***slender bar***. Each segment of ***length*** ℓ has ***mass*** m. There are six segments of length ℓ and one segment of length 2ℓ (segment 4). The mass center of the system is G and is located on the axis of rotation.

Reference frames:

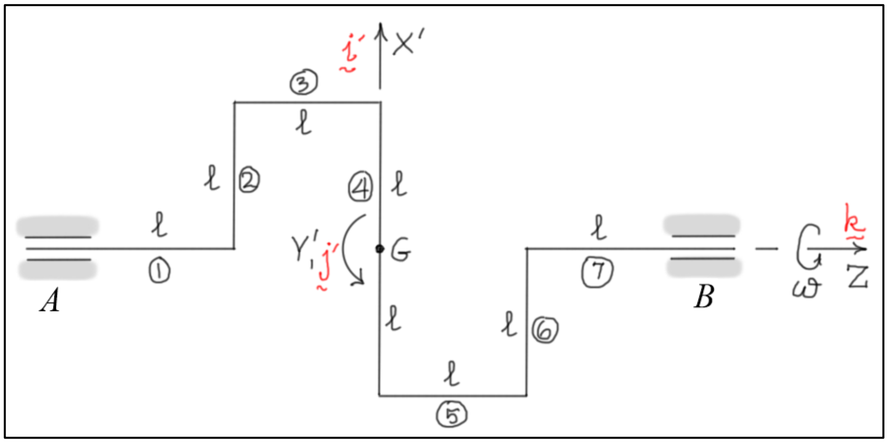
R: $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ (fixed frame)

S: $\underset{\sim}{i}', \underset{\sim}{j}', \underset{\sim}{k}$ (rotates with the shaft)

Find:

$\underset{\sim}{H}_G$ … angular momentum of the system about its mass center

K … kinetic energy of the system



Solution: (resolving the inertia and angular velocity about the shaft-fixed axes)

$$\begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{i}' \\ \underset{\sim}{H}_G \cdot \underset{\sim}{j}' \\ \underset{\sim}{H}_G \cdot \underset{\sim}{k} \end{Bmatrix} = \begin{bmatrix} I^G_{X'X'} & -I^G_{X'Y'} & -I^G_{X'Z} \\ -I^G_{Y'X'} & I^G_{Y'Y'} & -I^G_{Y'Z} \\ -I^G_{ZX'} & -I^G_{ZY'} & I^G_{ZZ} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix} = \begin{Bmatrix} -I^G_{X'Z}\omega \\ -I^G_{Y'Z}\omega \\ I^G_{ZZ}\omega \end{Bmatrix} \qquad (1)$$

where

$$I^G_{ZZ} = \sum_{i=1}^{7} \left(I^G_{ZZ}\right)_i = 0 + \tfrac{1}{3}m\ell^2 + m\ell^2 + \tfrac{1}{12}(2m)(2\ell)^2 + m\ell^2 + \tfrac{1}{3}m\ell^2 + 0 = \boxed{\tfrac{10}{3}m\ell^2}$$

$$I^G_{X'Z} = \sum_{i=1}^{7} \left(I^G_{X'Z}\right)_i = 0 + m(\tfrac{\ell}{2})(-\ell) + m(\ell)(-\tfrac{\ell}{2}) + 0 + m(-\ell)(\tfrac{\ell}{2}) + m(-\tfrac{\ell}{2})(\ell) + 0 = \boxed{-2m\ell^2}$$

$I^G_{Y'Z} \equiv 0$ ($X'Z$ plane is a plane of symmetry, so products associated with Y' are zero)

Substituting the inertias into Eq. (1) gives

$$\boxed{\underset{\sim}{H}_G = 2m\ell^2\omega\,\underset{\sim}{i}' + (\tfrac{10}{3})m\ell^2\omega\,\underset{\sim}{k}}$$

The ***kinetic energy*** of the crank shaft is found from the ***velocity*** and ***angular momentum*** vectors to be

$$K = \underbrace{\tfrac{1}{2}m({}^R\underset{\sim}{v}_G)^2}_{\text{zero}} + \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G = \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G = \tfrac{1}{2}(\omega\underset{\sim}{k}) \cdot \underset{\sim}{H}_G \quad \Rightarrow \boxed{K = \tfrac{10}{6}m\ell^2\omega^2}$$