Introductory Control Systems
Characteristics of Open-Loop and Closed-Loop Systems

Important Control System Characteristics

- ***Sensitivity*** of system response to ***parametric variations*** can be ***reduced***
- ***Transient*** and ***steady-state responses*** of a system can be ***altered***
- ***Steady-state error*** can be ***reduced***
- ***Response*** of system to ***disturbances*** (disturbance response) can be ***lowered***

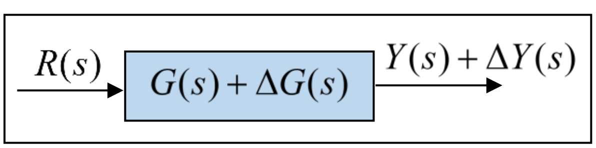
The effects of a control system on the overall response of a dynamic system can be ***positive*** or ***negative***. It is the ***responsibility*** of the ***analyst*** to ***design*** the control system so it has ***beneficial effects*** on system performance.

Parametric Sensitivity: Concept

From a ***mathematical perspective***, a dynamic system will have identical responses to repeated applications of the same input. The step response of a system as calculated by MATLAB, for example, is always the same (unless the transfer function is altered).

For ***real systems***, however, this is ***not*** the case. ***Each time*** a real system is subjected to an input, its response ***will vary***. The variations may be ***small random fluctuations*** producing the same average response, or they may be ***large fluctuations*** producing very different responses. For example, the effects of ***friction*** and ***damping*** can easily vary during the day-to-day operation of a system.

To ***compare*** the ***effects*** of ***variations*** on the response of ***open-loop*** and ***closed-loop*** systems, consider the systems shown in Fig. 1. In each system, the transfer function $G(s)$ has been ***changed*** to $G(s)+\Delta G(s)$. Here, $\Delta G(s)$ represents ***small changes*** to $G(s)$.



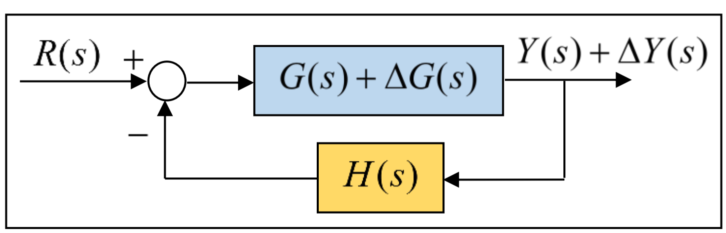


Fig. 1. Open-Loop and Closed-Loop Systems with Plant Variations

The ***algebraic equation*** associated with the ***open-loop system*** is

$$Y(s)+\Delta Y(s)=\left(G(s)+\Delta G(s)\right)R(s)=G(s)R(s)+\Delta G(s)R(s)$$

So, ***changes*** in the output can be written as

$$\boxed{\Delta Y(s)=\Delta G(s)R(s)} \qquad (1)$$

The plant changes are clearly ***passed directly to the system output***.

The ***algebraic equation*** associated with the ***closed-loop system*** is

$$\begin{aligned} Y(s)+\Delta Y(s) &= \left(\frac{G(s)+\Delta G(s)}{1+\left(G(s)+\Delta G(s)\right)H(s)}\right)R(s) \\ &= \left(\frac{G(s)}{1+\left(G(s)+\Delta G(s)\right)H(s)}\right)R(s)+\left(\frac{\Delta G(s)}{1+\left(G(s)+\Delta G(s)\right)H(s)}\right)R(s) \\ &\approx \left(\frac{G(s)}{1+GH(s)}\right)R(s)+\left(\frac{\Delta G(s)}{1+GH(s)}\right)R(s) \\ &\approx Y(s)+\left(\frac{\Delta G(s)}{1+GH(s)}\right)R(s) \end{aligned}$$

So, ***changes*** in the output can be approximated as

$$\boxed{\Delta Y(s)\approx\left(\frac{\Delta G(s)}{1+GH(s)}\right)R(s)} \qquad (2)$$

From this result it clear that the amount of $\Delta G(s)$ that is passed to the output depends on the magnitude of the ***loop*** (or open-loop) ***transfer function*** $GH(s)$. The ***larger*** the ***magnitude*** of $GH(s)$, the ***less*** changes in $G(s)$ will affect the system response.

Sensitivity: Calculation

The ***calculation*** of ***sensitivity*** is done more formally using derivatives. Specifically, S_α^T the sensitivity of a system with transfer function $T(s)$ to changes in a parameter α is defined as follows.

$$\boxed{S_\alpha^T=\frac{\alpha}{T}\left(\frac{\partial T}{\partial\alpha}\right)} \qquad (3)$$

Generally, the ***sensitivity*** is a function of s, and hence is a function of frequency.

If the ***magnitude*** of S_α^T is ***between zero and one*** $\left(0<\left|S_\alpha^T\right|<1\right)$, then the effects of $\Delta G(s)$ will be ***lowered*** (i.e. 10% changes in $G(s)$ will result in less than 10% changes in the response). However, if the ***magnitude*** of S_α^T is ***greater than one*** $\left(\left|S_\alpha^T\right|>1\right)$, then the effects of $\Delta G(s)$ will be ***magnified*** (i.e. 10% changes in $G(s)$ will result in greater than 10% changes in the response).

To gain some ***general insight*** into the issue of sensitivity for a simple closed-loop system (Fig. 2), consider the sensitivity of the system transfer function $T(s)=\dfrac{G}{1+GH}$ to ***bulk changes*** in the transfer functions $G(s)$ or $H(s)$.

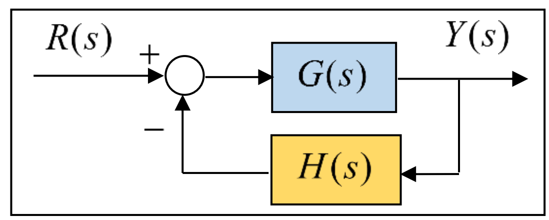


Fig. 2. Simple Closed Loop System

$$\boxed{S_G^T=\frac{G}{T}\left(\frac{\partial T}{\partial G}\right)} \quad \text{and} \quad \boxed{S_H^T=\frac{H}{T}\left(\frac{\partial T}{\partial H}\right)}$$

Using these ***definitions*** and the ***quotient rule*** for ***differentiation*** gives

$$S_G^T=\frac{G}{T}\frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right)=\cancel{G}\left(\frac{\cancel{1+GH}}{\cancel{G}}\right)\left(\frac{1\left(1+\cancel{GH}\right)-\cancel{GH}}{\cancel{(1+GH)}\,(1+GH)}\right)=\frac{1}{1+GH} \tag{4}$$

$$S_H^T=\frac{H}{T}\left(\frac{\partial T}{\partial H}\right)=H\left(\frac{\cancel{1+GH}}{\cancel{G}}\right)\left(\frac{-\cancel{G}G}{\cancel{(1+GH)}\,(1+GH)}\right)=\frac{-GH}{1+GH} \tag{5}$$

Eq. (4) indicates that as $\left|GH(s)\right|$ is ***increased***, the effects on the response of the system to changes in $G(s)$ are ***lowered***. This is the same conclusion that was drawn from Eq. (2). However, Eq. (5) indicates that as $\left|GH(s)\right|$ is ***increased***, the effects on the response of the system to changes in $H(s)$ are ***passed directly to the output***, that is, $S_H^T\approx 1$.

Note also that for an ***open-loop system*** with transfer function $T(s)=G(s)$,

$$\boxed{S_G^T=S_G^G=\frac{G}{G}\left(\frac{\partial G}{\partial G}\right)=1}$$

This again indicates that changes in the plant will be passed ***directly*** to the output.

Although Eqs. (4) and (5) provide ***general insights*** into the ***usefulness*** of ***closed-loop control***, Eq. (3) is used to determine the sensitivity to specific system parameters. For this reason, Eq. (3) provides more ***detailed information*** about the system at hand. For example, in previous notes, the ***closed-loop transfer function*** for ***proportional position control*** of a ***spring-mass-damper system*** was found to be

$$\boxed{T(s)=\frac{K}{ms^2+bs+(k+K)}}$$

The ***sensitivity*** of this system to changes in the damping and spring stiffness parameters can be calculated as follows.

$$S_b^T=\frac{b}{T}\frac{\partial T}{\partial b}=b\left(\frac{ms^2+bs+(k+K)}{K}\right)\frac{\partial}{\partial b}\left(\frac{K}{ms^2+bs+(k+K)}\right)$$

$$=b\left(\frac{\cancel{ms^2+bs+(k+K)}}{\cancel{K}}\right)\left(\frac{-\cancel{K}s}{\cancel{\left(ms^2+bs+(k+K)\right)}\left(ms^2+bs+(k+K)\right)}\right)$$

$$\Rightarrow\boxed{S_b^T=\frac{-bs}{ms^2+bs+(k+K)}}$$

$$S_k^T=\frac{k}{T}\frac{\partial T}{\partial k}=k\left(\frac{ms^2+bs+(k+K)}{K}\right)\frac{\partial}{\partial k}\left(\frac{K}{ms^2+bs+(k+K)}\right)$$

$$=k\left(\frac{\cancel{ms^2+bs+(k+K)}}{\cancel{K}}\right)\left(\frac{-\cancel{K}}{\cancel{\left(ms^2+bs+(k+K)\right)}\left(ms^2+bs+(k+K)\right)}\right)$$

$$\Rightarrow\boxed{S_k^T=\frac{-k}{ms^2+bs+(k+K)}}$$

<u>Control of Transient Response</u>

In previous notes the ***block diagram*** for the ***open-loop response*** of an ***armature-controlled DC motor*** was given. If the ***electrical response*** of the motor is much ***faster*** than the ***mechanical speed changes***, then the ***time dependence*** of the ***circuitry*** can be ***ignored***. Under these conditions, the block diagram reduces to that shown in Fig 3.

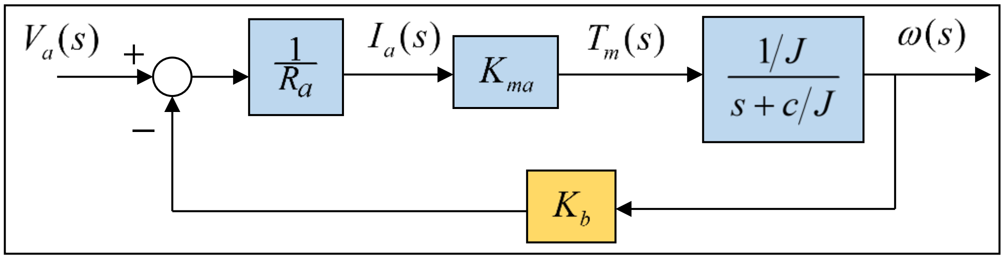


Fig. 3. Block Diagram of an Armature-Controlled DC Motor

As before, the ***input*** to the motor is the ***armature voltage*** $V_a(s)$ and the ***output*** is the ***angular velocity*** (speed) of the motor. Using block diagram reduction, the ***open-loop transfer function*** for this system is found to be

$$\frac{\omega(s)}{V_a(s)} = \frac{K^*}{s+a^*} \qquad \begin{cases} K^* = K_{ma}/R_a J \\ a^* = \left(R_a c + K_b K_{ma}\right)/R_a J \end{cases} \tag{6}$$

The parameters K^* and a^* are ***constants*** that depend on the ***motor characteristics***, ***inertial load***, and ***damping coefficient***. Note that the value of a^* ***determines how quickly the motor responds*** when a step increase in voltage is applied to the motor.

To study the ***effects of feedback*** on transient response, consider ***proportional, closed-loop control*** of the DC motor as shown in Fig 4. The ***input*** to the system is the ***desired angular velocity*** $\omega_d(s)$ and the ***output*** of the system is the ***actual angular velocity*** $\omega(s)$. The parameter K_t is the ***calibration constant of the tachometer*** that relates changes in angular velocity to changes in voltage. The signal $E(s)$ represents a tachometer voltage error, and the parameter K_a is the proportional gain.

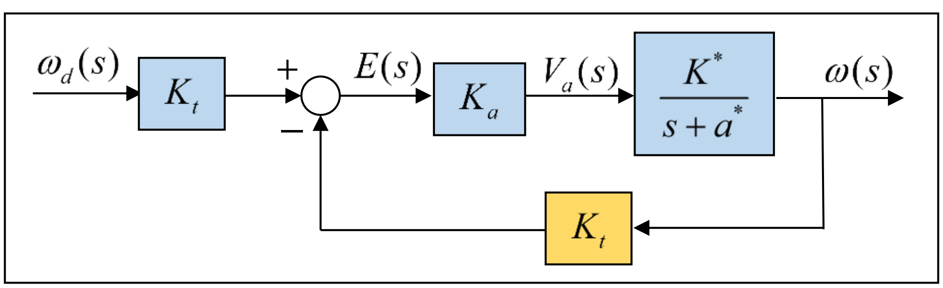


Fig. 4. Proportional Control of a DC Motor Using a Tachometer

The ***transfer function*** of the ***closed-loop system*** is found using block diagram reduction to be

$$\frac{\omega}{\omega_d}(s) = \frac{K_t \hat{K}}{s + \hat{a}} \quad \begin{cases} \hat{K} = K_a K^* \\ \hat{a} = a^* + K_a K^* K_t \end{cases} \tag{7}$$

The parameter $\hat{a}$ ***determines the speed of response*** of the ***closed-loop, speed control system***. The value of $\hat{a}$ can be ***increased*** by ***increasing*** the ***proportional gain*** K_a. Note, however, that if the value of K_a is increased ***too much***, the voltage input to the motor may become ***too large***, potentially damaging the motor. Physical limitations such as these are often not part of the mathematical model, so the analyst must be aware of them.

<u>Control of Steady-State Error</u>

Consider again the closed-loop speed control system of Fig. 4. To track the ***error*** in the system as it responds to a commanded speed change, the error signal $E(s)$ is taken to be the output of the system as shown in Fig. 5.

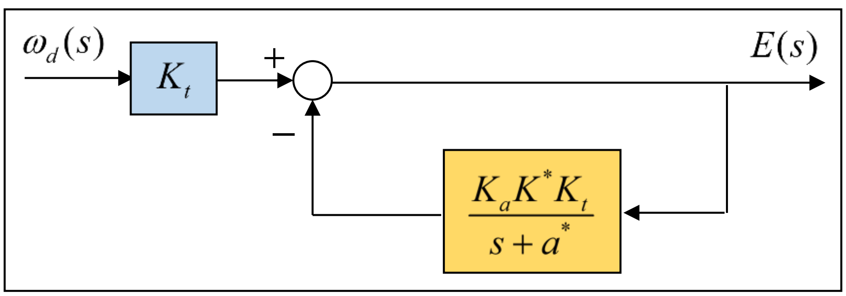


Fig. 5. Error of a DC Motor Speed Control System

Using block diagram reduction, the system ***error transfer function*** is found to be

$$\frac{E}{\omega_d}(s) = \frac{K_t(s + a^*)}{s + a^* + K_a K^* K_t} = \frac{K_t(s + a^*)}{s + \hat{a}} \tag{8}$$

Using the ***final value theorem***, the ***steady-state error*** to a unit step, speed change command is found to be

$$e_{ss} = \lim_{s \to 0}\left(\cancel{s} \cdot \frac{E}{\omega_d} \cdot \frac{1}{\cancel{s}} \right) = \frac{K_t a^*}{\hat{a}} \quad \text{(steady-state error)} \tag{9}$$

This result shows that the proportional control gain ***affects*** the steady-state error K_a. As the value of K_a is ***increased***, the value of $\hat{a}$ is ***increased*** and e_{ss} the ***steady-state error*** is ***decreased***.

Control of Disturbance Response

Consider the block diagram of an armature-controlled DC motor with a ***disturbance torque*** $T_D(s)$ as shown in Fig. 6. It is assumed that the disturbance torque ***reduces*** the torque generated by the motor under ideal conditions.

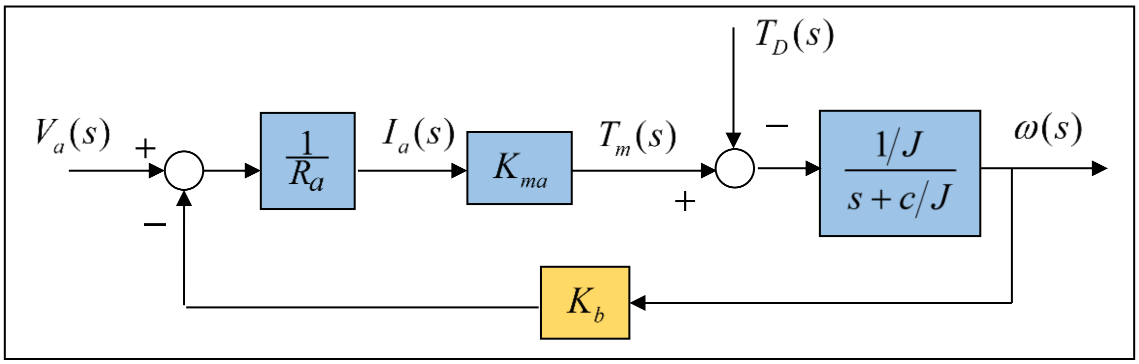


Fig. 6. Block Diagram of an Armature-Controlled DC Motor with Disturbance

To study the effect of the disturbance on the response of this system, the ***disturbance transfer function*** must be found. One way to do this is to move the disturbance to the left-most summing block as shown in Fig. 7.

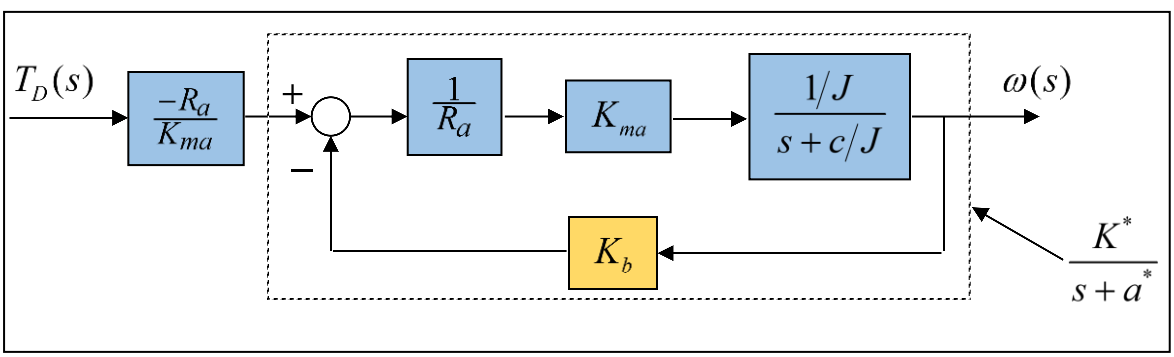


Fig. 7. Block Diagram of a DC Motor with Disturbance Input Only

The ***disturbance transfer function*** for the ***open-loop system*** is then identified to be

$$\boxed{\frac{\omega}{T_D}(s) = \frac{-R_a K^* / K_{ma}}{s + a^*}} \quad \text{(open loop system)} \tag{10}$$

Following this same approach, the ***disturbance transfer function*** of the ***closed-loop, speed control system*** of Fig. 4 can be found. In that case, to move the disturbance to the left-most summing block, the disturbance must be additionally moved over the proportional gain block as shown in Fig. 8.

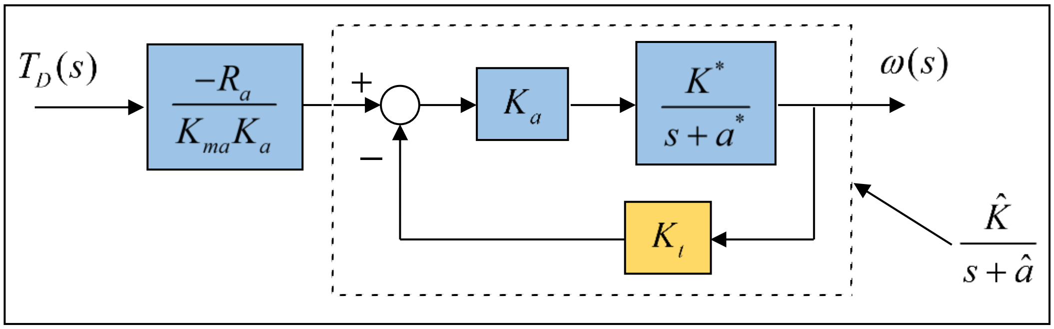


Fig. 8. Disturbance Input in Speed Control System of a DC Motor

Using block diagram reduction, the ***disturbance transfer function*** for the ***closed-loop, speed control system*** is found to be

$$\frac{\omega}{T_D}(s)=\frac{-R_a\hat{K}/K_{ma}K_a}{s+\hat{a}}=\frac{-1/J}{s+\hat{a}} \quad \text{(closed-loop, speed control system)} \qquad (11)$$

The ***steady-state angular velocity change*** of the motor to a ***unit step disturbance torque*** is found using the final value theorem.

$$\left(\omega_{ss}\right)_{T_D}=\lim_{s\to 0}\left(\cancel{s}\cdot\frac{\omega}{T_D}\cdot\frac{1}{\cancel{s}}\right)=\frac{-1}{J\hat{a}} \quad \text{(closed-loop, speed control system)} \qquad (12)$$

These last two results indicate that as the proportional gain K_a is ***increased***, the disturbance response ***decays faster***, and the ***steady-state angular velocity change*** is ***decreased***. Both are positive effects.