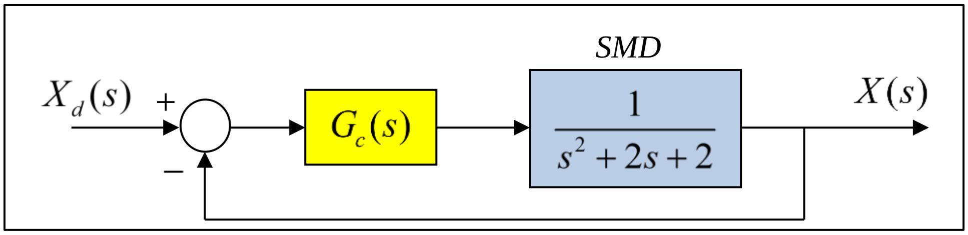
Introductory Motion and Control
Root Locus Design of a Phase-Lag Compensator for a Spring-Mass-Damper (SMD) Positioning System

To illustrate the ***root locus design*** of a ***phase-lag compensator***, consider the following SMD positioning system controlled by the compensator $G_c(s)$. Here, $X_d(s)$ and $X(s)$ are the ***desired*** and ***actual*** positions of the mass.



Using proportional control ($G_c(s)=K$), ***large*** gains are required to control ***steady-state error*** for a step input. Unfortunately, ***large gains*** produce ***undesirable, oscillatory*** closed-loop response. Below, a phase-lag compensator is designed to lower the steady-state error without introducing highly oscillatory behavior.

Problem: Design a phase-lag compensator so the closed-loop system has a ***steady-state step position error*** $e_{ss}=1-x_{ss}<0.1$ and ***damping factors*** for the complex poles of $\zeta \geq 0.4$. Plot the step response of the resulting closed-loop system.

Root Locus Design

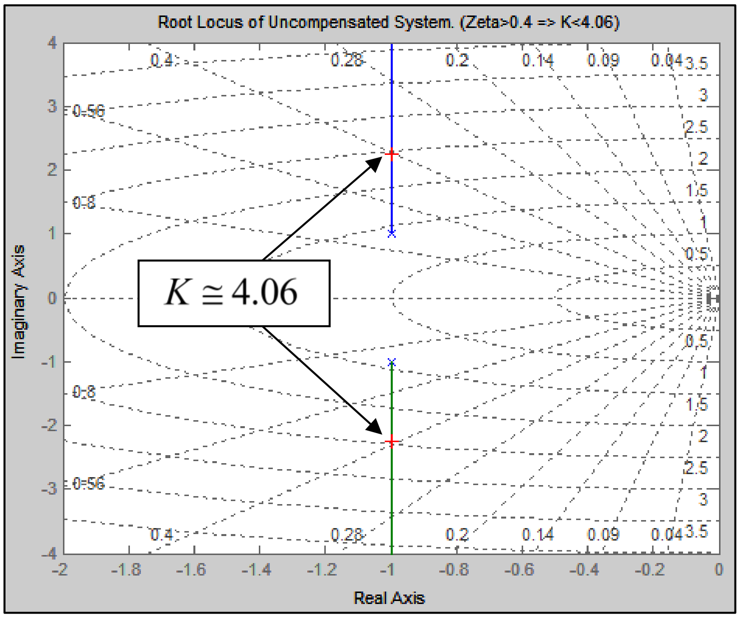
Step 1: Examine ***RL diagram*** of ***uncompensated system.*** Find the ***gain*** required to satisfy the ***steady-state error*** requirement.

The root locus diagram of the uncompensated system is very simple. The poles of $GH(s)$ are $-1\pm 1j$. For $K>0$ the roots move to infinity along the asymptotes at $\sigma_A=-1$. For $K<0$ the roots move into the break point at –1 and then move along the positive and negative real axis.

For $\zeta \geq 0.4$, we use MATLAB to show that $K \leq 4.06$. See diagram below. Using $K=4.06$, the steady-state error for the uncompensated system is

$$e_{ss}=\lim_{s\to 0}\left[\frac{1}{1+GH(s)}\right]=\frac{1}{1+(4.06/2)}=0.33$$

This is clearly higher than the specified value of 0.1. To lower the steady-state error to 0.1, we require $\boxed{K=18}$.



Root Locus of Uncompensated System. For the damping ratio to be greater than 0.4, the system gain must be less than 4.06.

If $K = 18$ is used, a ***phase-lag compensator*** can be added to increase the system damping. This will also ***increase the settling time*** of the system.

Step 2: Evaluate how the ***compensator*** changes the ***RL diagram.***

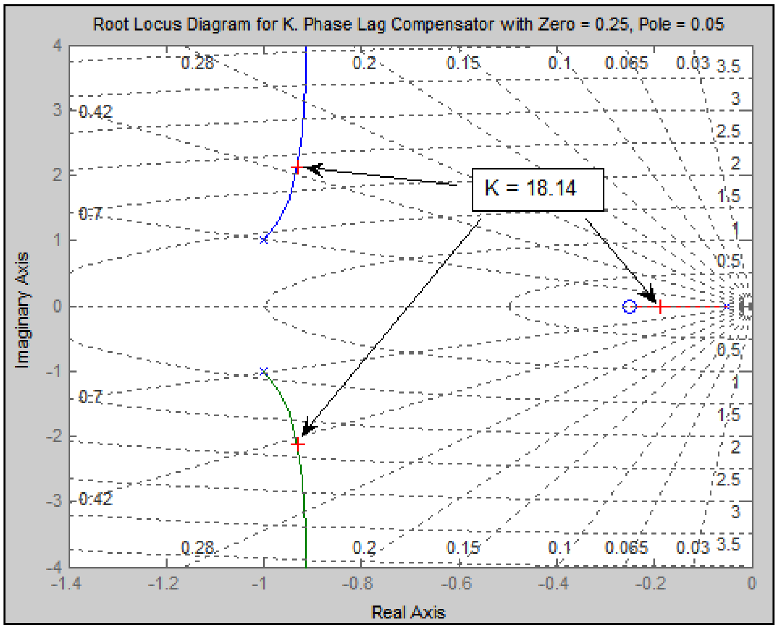
Here, the loop transfer function $GH(s) = \dfrac{K\alpha(s+z)}{(s+p)(s^2+2s+2)}$, so the system has asymptotes at $\phi_A = \pm 90$ (deg) that intersect the real axis at $\sigma_A = \dfrac{2(-1)-p+z}{2}$. In a phase-lag compensator, $z > p$, so the asymptotes will be ***moved to the right***. However, they will not be moved far, because the ***pole*** and ***zero*** are generally located ***close to each other***.

Step 3: Try ***different pole-zero combinations*** to see effect on RL diagram

Based on the ratio of the uncompensated system gain $(K \approx 4)$ and the desired gain $(K \approx 18)$, start by picking a pole close to the origin (say $p = 0.1$) and setting $\boxed{\alpha = K_{\text{uncompensated}} / K_{\text{desired}} = 4/18 \approx 0.2}$ and $\boxed{z = p/\alpha = 0.1/0.2 = 0.5}$.

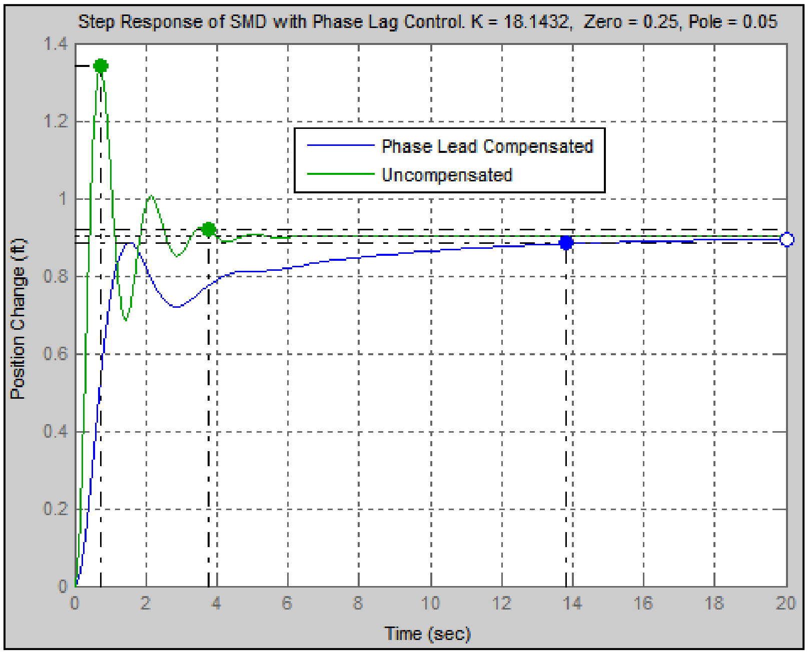
Two questions must now be answered: 1) Can roots be found with $\zeta > 0.4$? 2) Can a ***large enough value for*** K be chosen so the steady-state error is small? From the root locus diagram, if $\zeta \geq 0.4$, then the maximum K value is approximately 16. ***After some iteration*** (moving the pole/zero combination closer to the imaginary axis), the following results were found.

For $p = 0.05$ and $z = 0.25$, the loop transfer function of the compensated system is $GH(s) = \dfrac{0.2K(s+0.25)}{(s+0.05)(s^2+2s+2)}$, and the root locus diagram yields $K \approx 18$ when the complex poles have $\zeta \approx 0.4$. See the diagram below.



Step 4: Check the step response.

The step response of the ***uncompensated system*** with gain $K \approx 18.14$ (same steady-state error as the compensated system) shows a ***large overshoot*** (49%), low damping, and a ***settling time*** of approximately 3.8 seconds, while the step response of the compensated system shows no overshoot, higher damping, and a settling time around 13.8 seconds.



If some ***overshoot is acceptable*** (allowing smaller ζ), the ***gain can be increased*** in the above design to yield the results shown below for a gain of $K \approx 27.44$. With this gain the system has about 10% overshoot and a settling time of 11 seconds. Note also that with the larger gain, the steady state error is smaller ($e_{ss} \approx 0.07$).