

Intermediate Dynamics

Angular Momentum and Kinetic Energy of Example System II

Intermediate Dynamics – Example #13:

Reference frames: (R is the fixed frame)

$F: (\underline{n}_1, \underline{n}_2, \underline{k}) \dots$ (rotates with frame F)

$B: (\underline{e}_1, \underline{n}_2, \underline{e}_3) \dots$ (rotates with the bar B)

Find:

$\underline{H}_G \dots$ **angular momentum** of B about its mass center, G

$K \dots$ **kinetic energy** of B

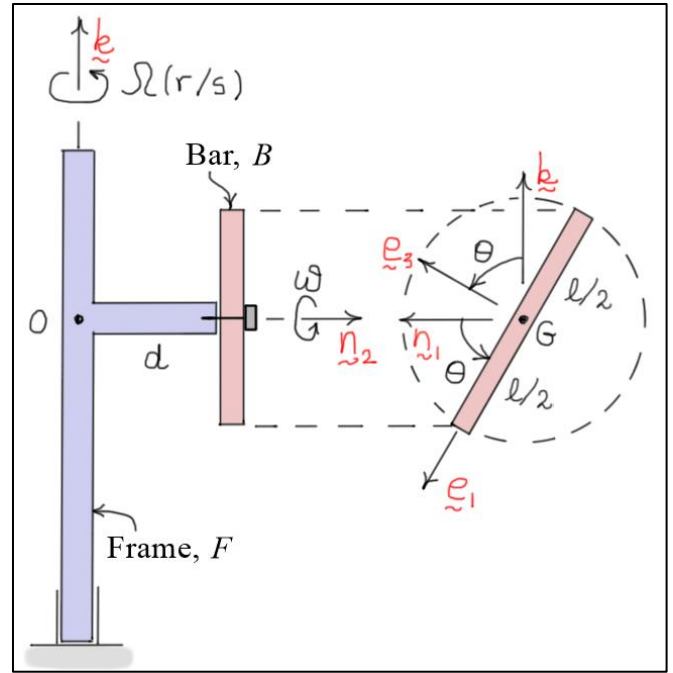
Solution:

The **angular velocity vector** and the **inertia matrix** about **body-fixed axes** are:

$$[I_G]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}m\ell^2 & 0 \\ 0 & 0 & \frac{1}{12}m\ell^2 \end{bmatrix}$$

$$\text{and } {}^R\omega_B = \omega \underline{n}_2 + \Omega \underline{k} = \omega \underline{n}_2 + \Omega(-S_\theta \underline{e}_1 + C_\theta \underline{e}_3)$$

$$\Rightarrow {}^R\omega_B = (-\Omega S_\theta) \underline{e}_1 + \omega \underline{n}_2 + (\Omega C_\theta) \underline{e}_3$$



Using these results, the **body-fixed components** of the **angular momentum** vector are

$$\begin{Bmatrix} \underline{H}_G \cdot \underline{e}_1 \\ \underline{H}_G \cdot \underline{n}_2 \\ \underline{H}_G \cdot \underline{e}_3 \end{Bmatrix} = \frac{m\ell^2}{12} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -\Omega S_\theta \\ \omega \\ \Omega C_\theta \end{Bmatrix} = \frac{m\ell^2}{12} \begin{Bmatrix} 0 \\ \omega \\ \Omega C_\theta \end{Bmatrix} \Rightarrow \underline{H}_G = \frac{m\ell^2}{12} (\omega \underline{n}_2 + \Omega C_\theta \underline{e}_3)$$

Using these results, \underline{H}_G can also be expressed using unit vectors fixed in F .

$$\underline{H}_G = \frac{m\ell^2}{12} (\omega \underline{n}_2 + \Omega C_\theta \underline{e}_3) = \frac{m\ell^2}{12} [\omega \underline{n}_2 + \Omega C_\theta (S_\theta \underline{n}_1 + C_\theta \underline{k})]$$

$$\Rightarrow \underline{H}_G = \frac{m\ell^2}{12} [\Omega C_\theta S_\theta \underline{n}_1 + \omega \underline{n}_2 + \Omega C_\theta^2 \underline{k}]$$

Using the body-fixed components of \underline{H}_G and ${}^R\omega_B$, the kinetic energy of the bar can now be calculated as follows.

$$K = K_{translation} + K_{rotation} = \frac{1}{2}m({}^R\underline{v}_G)^2 + \frac{1}{2}({}^R\omega_B \cdot \underline{H}_G)$$

$$= \frac{1}{2}md^2\Omega^2 + \frac{1}{2}(-\Omega S_\theta \underline{e}_1 + \omega \underline{n}_2 + \Omega C_\theta \underline{e}_3) \cdot \frac{m\ell^2}{12} (\omega \underline{n}_2 + \Omega C_\theta \underline{e}_3)$$

$$\Rightarrow K = \frac{1}{2}md^2\Omega^2 + \frac{m\ell^2}{24} (\omega^2 + \Omega^2 C_\theta^2)$$