Intermediate Dynamics
Newton/Euler Equations of Motion for a Rigid Body

Using the theory of ***systems of particles***, it can be shown that the ***equations of motion*** for rigid body motion in an ***inertial frame*** R can be written as follows.

$$\boxed{\begin{aligned} \sum_i \underset{\sim}{F}_i &= m\,{}^R\underset{\sim}{a}_G \\ \sum_i \left(\underset{\sim}{M}_G\right)_i &= \frac{{}^R d}{dt}\left(\underset{\sim}{H}_G\right) \end{aligned}} \qquad (1)$$

Here, ${}^R\underset{\sim}{a}_G$ is the ***acceleration*** of G the ***mass-center*** of the body, and $\underset{\sim}{H}_G = \underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B$ is the ***angular momentum*** of the body about its mass-center. Using the "***derivative rule***" the right side of the moment equation can be rewritten as follows.

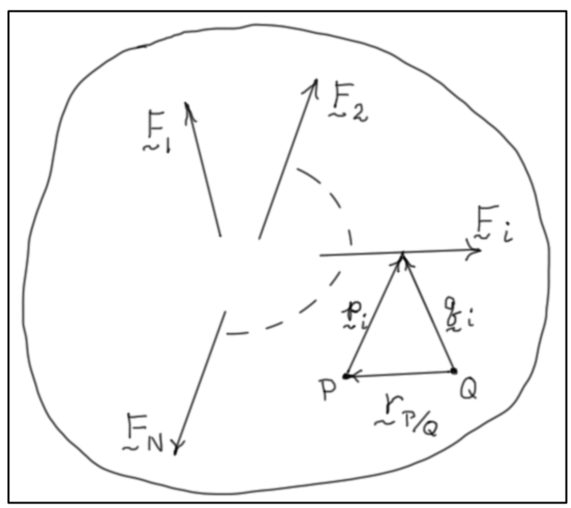
$$\sum_i \left(\underset{\sim}{M}_G\right)_i = \frac{{}^R d}{dt}\left(\underset{\sim}{H}_G\right) = \frac{{}^R d}{dt}\left(\underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B\right) = \frac{{}^B d}{dt}\left(\underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)$$

or

$$\boxed{\sum_i \left(\underset{\sim}{M}_G\right)_i = \left(\underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\alpha}_B\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)} \qquad (2)$$

Equivalent Force Systems

The moment equation (Eq. (2)) can be extended to ***taking moments about any point*** using the concept of ***equivalent force systems***. Systems of forces are said to be ***equivalent*** if they have the ***same resultant*** and they have the ***same moment about any point***. The ***resultant*** $\underset{\sim}{R}$ of a force system is simply the ***sum of all the forces***.



$$\boxed{\underset{\sim}{R} = \sum_i \underset{\sim}{F}_i}$$

The ***moment*** of the system about some point P is

$$\boxed{\sum_i \left(\underset{\sim}{M}_P\right)_i = \sum_i \left(\underset{\sim}{p}_i \times \underset{\sim}{F}_i\right)}$$

Finally, note that the ***moment*** of the system about ***another point*** Q can be related to the moment about P as follows.

$$\sum_i \left(\underset{\sim}{M}_Q\right)_i = \sum_i \left(\underset{\sim}{q}_i \times \underset{\sim}{F}_i\right) = \sum_i \left(\left[\underset{\sim}{r}_{P/Q} + \underset{\sim}{p}_i\right] \times \underset{\sim}{F}_i\right) = \sum_i \left(\underset{\sim}{r}_{P/Q} \times \underset{\sim}{F}_i\right) + \sum_i \left(\underset{\sim}{p}_i \times \underset{\sim}{F}_i\right)$$

$$= \underset{\sim}{r}_{P/Q} \times \left(\sum_i \underset{\sim}{F}_i\right) + \sum_i \left(\underset{\sim}{M}_P\right)_i$$

or

$$\boxed{\sum_i \left(\underset{\sim}{M}_Q\right)_i = \sum_i \left(\underset{\sim}{M}_P\right)_i + \underset{\sim}{r}_{P/Q} \times \left(\sum_i \underset{\sim}{F}_i\right)}$$

Alternate Moment Equation

In the above analysis, let Q be any point A, and let P be the mass center G. Then using the relationship between the sum of the moments about different points, we can write

$$\sum_i \left(\underset{\sim}{M}_A\right)_i = \sum_i \left(\underset{\sim}{M}_G\right)_i + \left(\underset{\sim}{r}_{G/A} \times m\, {}^R\underset{\sim}{a}_G\right)$$

or

$$\boxed{\sum_i \left(\underset{\sim}{M}_A\right)_i = \left(\underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\alpha}_B\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right) + \left(\underset{\sim}{r}_{G/A} \times m\, {}^R\underset{\sim}{a}_G\right)} \quad (A \text{ is any point})$$

Special Case: Motion about a Fixed Point

If some point O of the body is fixed so that the body pivots about that point, the above equations of motion can be shown to take the form

$$\boxed{\begin{aligned} &\sum_i \underset{\sim}{F}_i = m\, {}^R\underset{\sim}{a}_G \\ &\sum_i \left(\underset{\sim}{M}_O\right)_i = \frac{{}^R d}{dt}\left(\underset{\sim}{H}_O\right) = \left(\underset{\approx}{I}_O \cdot {}^R\underset{\sim}{\alpha}_B\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_O\right) \end{aligned}}$$

where $\underset{\sim}{H}_O = \underset{\approx}{I}_O \cdot {}^R\underset{\sim}{\omega}_B$ is the ***angular momentum*** of the body about the ***fixed-point*** O. Note that the elements of the inertia dyadic $\underset{\approx}{I}_O$ can be determined using the ***parallel axes theorems*** for moments and products of inertia.

Note: If the expressions used in these equations are ***valid only at some instant of time***, then the equations are ***algebraic***. If the expressions are ***valid for all time***, then the equations are ***differential equations*** and can be ***integrated numerically*** to simulate the motion of the system.