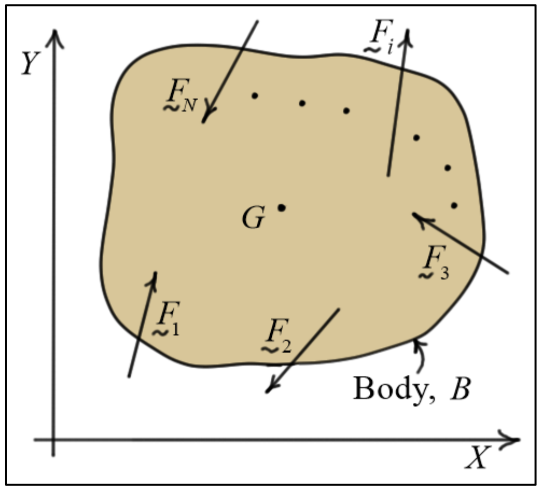
Elementary Dynamics
Work and Energy Principles for Rigid Body Motion in Two Dimensions

Principle of Work and Energy for a Single Body

The figure shows a rigid body undergoing planar motion under the action of forces $\underset{\sim}{F}_i \ (i=1,\ldots,N)$. As a result of work done on the body by all the forces and torques (equal and opposite forces), it will experience changes in ***kinetic energy***. The ***principle*** of ***work*** and ***energy*** states



$$\boxed{KE_1 + U_{1\to 2} = KE_2}$$

Here, $U_{1\to 2}$ represents the ***work done*** by all the forces and torques acting on the body as it moves from position 1 to position 2, and $KE_i \ (i=1,2)$ represent the kinetic energies of the body in those two positions.

The fundamental definition of kinetic energy of a rigid body is

$$KE = \int_B \tfrac{1}{2}(\underset{\sim}{v}_P \cdot \underset{\sim}{v}_P)\,dm$$

Here, $\underset{\sim}{v}_P$ represents the velocity of a generic point P within the body. If the body is undergoing ***general plane motion***, it can be shown that

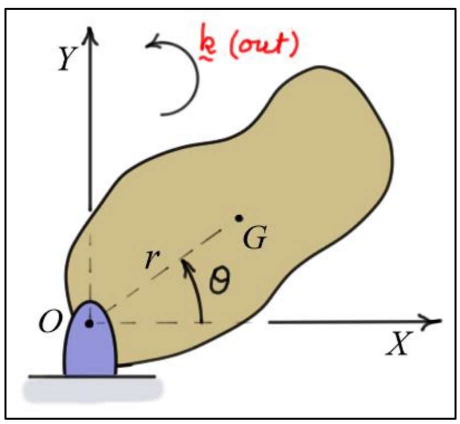
$$\boxed{KE = (KE)_{\text{translation}} + (KE)_{\text{rotation}} = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2} I_G \omega^2} \qquad \text{(general plane motion)}$$

Here, v_G^2 is the square of the mass-center velocity, ω^2 is the square of the body's angular velocity, and I_G is the mass moment of inertia of the body about its mass-center.

If the body is undergoing ***fixed-axis rotation*** about point O, the kinetic energy can be rewritten in a simpler form as

$$\boxed{KE = \tfrac{1}{2} I_O \omega^2} \qquad \text{(fixed-axis rotation)}$$

Here, I_O is the moment of inertia of the body about the axis of rotation.



Principle of Work and Energy for Systems of Bodies

The principle of work and energy can also be applied to a ***system*** of ***bodies***. As for a single body, the principle states

$$\boxed{KE_1 + U_{1\to 2} = KE_2}$$

Here, $U_{1\to 2}$ represents the work done by all the forces and torques acting on the ***system*** of ***bodies*** as they move from position 1 to position 2, and $KE_i \ (i = 1, 2)$ represent the kinetic energies of the ***system*** of ***bodies*** in those positions. The kinetic energy of a system of bodies at any time is simply the ***sum*** of the kinetic energies of the individual bodies of the system at that time

$$\boxed{KE = \sum_{\substack{\text{bodies} \\ (i)}} \left(\tfrac{1}{2} m v_{G_i}^2 + \tfrac{1}{2} I_G \omega_i^2\right)}$$

It is important when applying the principle of work and energy to a system of bodies to know which forces and torques ***do work*** and ***which do not***.

Work Done by External Forces on the System

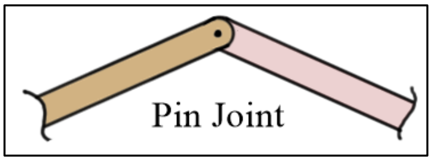
Conservative Forces:

Conservative forces applied to the system at locations that have ***nonzero displacement*** as the system moves from position 1 to position 2 do ***nonzero work***. Conversely, conservative forces applied to the system at locations that have ***zero net displacement*** as the system moves from position 1 to position 2 do ***zero (no) work***.

Nonconservative Forces:

Nonconservative forces applied to the system at locations that have ***nonzero displacement*** as the system moves from position 1 to position 2 do ***nonzero work***. Nonconservative forces applied to the system at locations that move but have ***zero net displacement*** as the system moves from position 1 to position 2 also do ***nonzero work***. ***Nonconservative forces do no work only when they are applied at points that do not move.***

Work Done by Internal Pin Forces



As stated above, the work done by forces at fixed support points do no work, because those points do not move. The work done by forces at pin joints that move is ***nonzero*** on each of the members. However, the ***net work*** done on the system of the two members is ***zero***, because the pin forces on the members are equal and opposite and move through the same displacement.

Work Done by External Torques

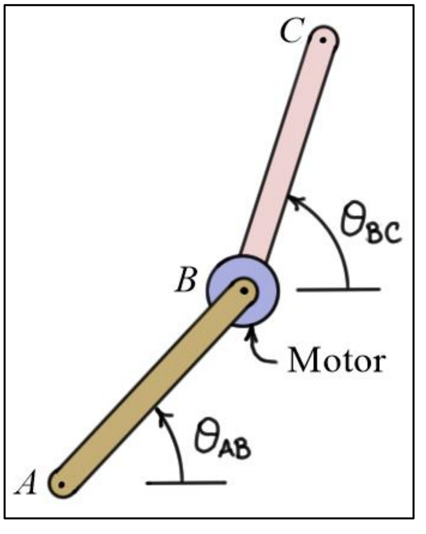
Conservative Torques:

Conservative torques applied to the system at locations having ***nonzero rotation*** as the system moves from position 1 to position 2 do ***nonzero work***. Conversely, conservative torques applied to the system at locations that have ***zero net rotation*** as the system moves from position 1 to position 2 do ***zero*** **(*no*)** ***work***.

Nonconservative Torques:

Nonconservative torques applied to the system at locations that have ***nonzero rotation*** as the system moves from position 1 to position 2 do ***nonzero work***. Nonconservative torques applied to the system at locations that rotate but have ***zero net rotation*** as the system moves from position 1 to position 2 also do ***nonzero work***. ***Nonconservative torques do no work only when they are applied at points that do not rotate.***

Work Done by Torques Through a Pin Joint



The work done by a torque acting at a pin joint between two bodies is generally ***nonzero***. As an example, consider a motor located between two segments of a robot arm as shown. The net work done by the motor on the arm is the sum of the work done on each of the segments (*AB* and *BC*)

$$U_{1\to 2} = \int_{(\theta_{AB})_1}^{(\theta_{AB})_2} (-M)\,d\theta + \int_{(\theta_{BC})_1}^{(\theta_{BC})_2} (M)\,d\theta \neq 0$$

Note if the motor torque is ***constant***, the above equation reduces to

$$\boxed{U_{1\to 2} = M(\Delta\theta_{BC} - \Delta\theta_{AB})}$$

Work Done by Conservative Forces and Torques

If a force or torque is ***conservative*** (e.g. springs and gravity), $U_{1\to 2}$ can be calculated using a ***potential energy*** function.

$$\boxed{U_{1\to 2} = V_1 - V_2}$$

Here,

$$\boxed{V_{\substack{\text{translational}\\ \text{spring}}} = \tfrac{1}{2} k_t e^2} \qquad \boxed{V_{\substack{\text{rotational}\\ \text{spring}}} = \tfrac{1}{2} k_\theta \theta^2} \qquad \boxed{V_{\text{gravity}} = mgh_G}$$

Here, e represents the change in length (either increase or decrease) of the spring from its natural length, θ represents the change in angle (either increase or decrease) of the spring from its undeformed angle, and h_G represents the distance G the mass center of the body is above some ***fixed*** and ***arbitrary datum***.

The units of potential energy are the same as those for work: foot-pounds or Newton-meters. Hence, the units of coefficient k_t are pounds per foot or Newtons per meter, and the units for coefficient k_θ are foot-pounds or Newton-meters (assuming θ is in radians).

Principle of Conservation of Energy

If all the forces and torques acting on a system are ***conservative***, the principle of work and energy can be written as

$$KE_1 + U_{1\to 2} = KE_1 + V_1 - V_2 = KE_2$$

or

$$\boxed{KE_1 + V_1 = KE_2 + V_2 = \text{constant}}$$

Here, $KE + V$ represents the total mechanical energy of the system.