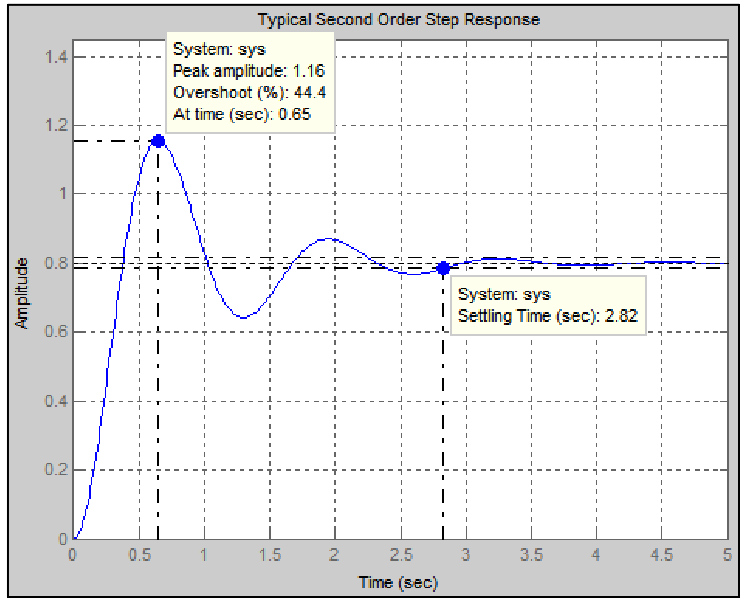
Introductory Control Systems
Characteristics of Under-damped, Second-Order System Step Response
Ref: Dorf & Bishop, Modern Control Systems, 12th edition, Prentice-Hall, Inc, 2010

The general form of the transfer function of an ***under-damped***, ***second-order system*** with no zero (constant numerator) is

$$\frac{Y}{R}(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



For example, if $K = 20$, $\omega_n = 5\,(\text{r/s})$, and $\zeta = 0.25$, the ***unit step response*** of the system is as shown in the figure. The MATLAB plot indicates there is approximately a 44% overshoot, a settling time $T_s \approx 2.8\,(\text{sec})$, and a final value $f_v = 0.8$.

Using the table of Laplace transforms, it can be shown that the ***response function*** can be written as

$$y(t) = \frac{K}{\omega_n^2}\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right)\right)$$

From this result, relatively simple formulae can be derived for the basic ***characteristics*** of the response curve. The ***percent overshoot*** and the ***peak time*** (time it takes for the system to reach the top of the first peak) are given by the formulae

$$\%OS = 100 e^{-\zeta\pi/\sqrt{1-\zeta^2}} \qquad T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Values of $\%OS$ and $\omega_n T_p$ can also be read from Fig. 5.8 from the Dorf & Bishop text. The settling time for the system is ***approximately*** $T_s \approx 4/\zeta\omega_n = 3.2\ (\text{sec})$. Control system engineers often refer also to the ***rise time*** (T_r) of a system. This is often defined as the time it takes the system to go from 10% to 90% of the final value for the first time. In the above example, $T_r \approx 0.25\ (\text{sec})$.

For second order systems that ***have a zero***, the transient response can be affected by the ***location*** of the zero. If the zero is ***far left*** of the poles (along the real axis), then the above formula for percent overshoot can be used; however, if the zero is ***close to*** or ***to the right of*** the poles, then the above formula will under-predict the percent overshoot. The amount of error in using the equation can be large.