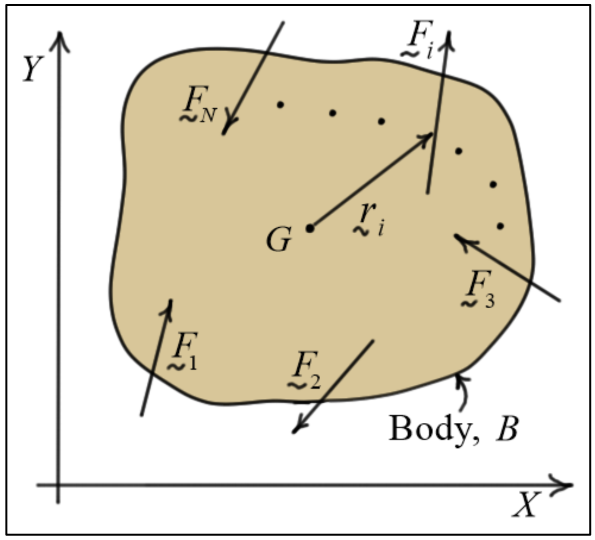
Elementary Dynamics
Principle of Impulse and Momentum for Rigid Body Motion in Two Dimensions

General Plane Motion



The figure shows a rigid body moving in two dimensions. The motion is caused by a series of N forces $\underset{\sim}{F}_i\ (i=1,\ldots,N)$. Generally, each force has the effect of ***translating*** and ***rotating*** the body. Newton's laws of translational and rotational motion are

$$\sum_{i=1}^{N} \underset{\sim}{F}_i = m\,\underset{\sim}{a}_G = m\left(\frac{d\underset{\sim}{v}_G}{dt}\right) = \frac{d}{dt}(m\,\underset{\sim}{v}_G)$$

$$\sum_{i=1}^{N} (\underset{\sim}{M}_G)_i = \sum_{i=1}^{N} (\underset{\sim}{r}_i \times \underset{\sim}{F}_i) = I_G\,\underset{\sim}{\alpha} = I_G\left(\frac{d\underset{\sim}{\omega}}{dt}\right) = \frac{d}{dt}(I_G\,\underset{\sim}{\omega})$$

Note here that $(\underset{\sim}{M}_G)_i$ represents the ***moment*** of force $\underset{\sim}{F}_i$ about the mass center G. Also, recall that I_G represents the ***mass moment of inertia*** of the body about a Z axis passing through the mass center G.

The above equations can be ***integrated with respect to time*** to give

$$\boxed{\left(m\underset{\sim}{v}_G\right)_1 + \int_{t_1}^{t_2}\left(\sum_{i=1}^{N}\underset{\sim}{F}_i\right)dt = \left(m\underset{\sim}{v}_G\right)_2} \qquad \text{(Principle of Linear Impulse \& Momentum)}$$

$$\boxed{\left(I_G\,\underset{\sim}{\omega}\right)_1 + \int_{t_1}^{t_2}\left(\sum_{i=1}^{N}(\underset{\sim}{r}_i \times \underset{\sim}{F}_i)\right)dt = \left(I_G\,\underset{\sim}{\omega}\right)_2} \qquad \text{(Principle of Angular Impulse \& Momentum)}$$

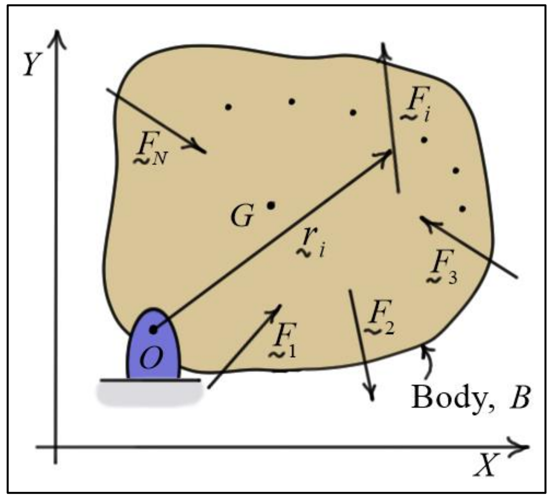
The ***principle of linear impulse and momentum*** states that the linear impulses applied to the body over the ***time interval*** $t_1 \to t_2$ give rise to a ***change*** in the linear momentum of the body. The ***principle of angular impulse and momentum*** states that the angular impulses applied to the body over the ***time interval*** $t_1 \to t_2$ give rise to a ***change*** in the angular momentum of the body. Note the linear momentum is often written as $\underset{\sim}{L} = m\underset{\sim}{v}_G$, and the angular momentum about G the mass center as $\underset{\sim}{H}_G = I_G\,\underset{\sim}{\omega}$.

Note that, like Newton's laws of translational and rotational motion, the above equations are ***vector equations***. For planar motion, there are ***two scalar linear momentum equations*** (X and Y directions) and ***one scalar angular momentum equation*** (Z direction).

Special Case

Fixed Axis Rotation

When a body is undergoing ***fixed axis rotation*** as shown, the principle of angular momentum can be written about the fixed-point O as

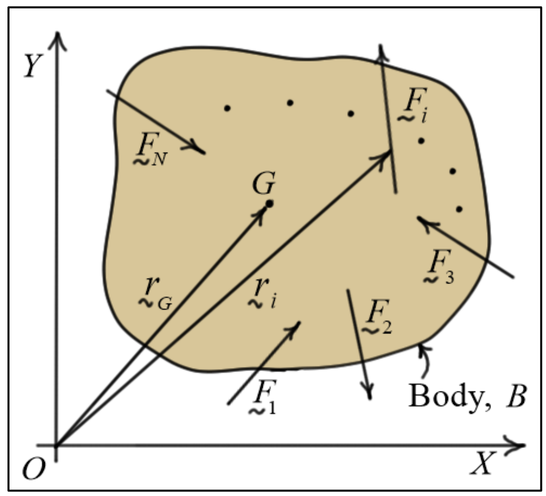


$$\left(I_O \underset{\sim}{\omega}\right)_1 + \int_{t_1}^{t_2}\left(\sum_{i=1}^{N}(\underset{\sim}{r}_i \times \underset{\sim}{F}_i)\right)dt = \left(I_O \underset{\sim}{\omega}\right)_2$$

The term $\sum_{i=1}^{N}(\underset{\sim}{r}_i \times \underset{\sim}{F}_i)$ represents the moment of all the forces about the fixed point O, and I_O represents the mass moment of inertia of the body about a Z axis passing through O.

Principle of Angular Momentum about an Arbitrary Fixed Point

Consider a rigid body undergoing planar motion. The principle of angular impulse and momentum can also be written relative to an arbitrary fixed-point O as follows



$$(\underset{\sim}{H}_O)_1 + \int_{t_1}^{t_2}\left(\sum_{i=1}^{N}(\underset{\sim}{r}_i \times \underset{\sim}{F}_i)\right)dt = (\underset{\sim}{H}_O)_2$$

Here,

$$\underset{\sim}{H}_O = I_G\,\underset{\sim}{\omega} + (\underset{\sim}{r}_G \times m\underset{\sim}{v}_G)$$

Note that $\underset{\sim}{H}_O$ is the sum of the angular momentum of the body about G and the moment of the linear momentum about O. The linear momentum $\underset{\sim}{L} = m\underset{\sim}{v}_G$ is assumed to have a line of action through G.