Introductory Control Systems
Second-Order System Step Response – Summary

Ref: R. N. Clark, *Introduction to Automatic Control Systems*, John Wiley & Sons, Inc, 1962
Dorf & Bishop, *Modern Control Systems*, 12th edition, Prentice-Hall, Inc, 2010

The transfer functions for second-order systems can be written in one of the two general forms (depending on whether the system has a zero or not)

Case 1: $\boxed{\frac{Y}{R}(s)=\frac{q}{s^2+ps+q}}$ Case 2: $\boxed{\frac{X}{R}(s)=\frac{(q/a)(s+a)}{s^2+ps+q}}$

Both cases can be broken into different types of response depending on whether the poles of the system are ***real and unequal***, ***real and equal***, ***complex***, or ***purely imaginary***. The discussion below considers only the response of ***stable*** systems. Stable systems (as defined here) are systems whose poles have ***non-positive*** real parts.

Case 1: There are ***four types of motion*** that are possible in this case. $\boxed{\frac{Y}{R}(s)=\frac{q}{s^2+ps+q}}$

- If the poles are ***real and unequal***, the response is ***over-damped*** with ***no overshoot***. If the poles are ***widely separated***, the response may be ***dominated*** by the smaller (***slower***) pole.
- If the poles are ***real and equal***, the response is ***critically damped*** with ***no overshoot***.
- If the poles are ***complex*** the system is ***under-damped*** with an overshoot.
- If the poles are ***purely imaginary***, then the system has ***no damping*** and the response will be ***oscillatory*** with no reduction in amplitude. The response is said to be harmonic.

Fig. 1 shows the step response of a system with ***real, unequal*** poles, and Fig. 2 shows the step response of a system with ***complex*** poles. Both plots show the 2% ***settling time*** of the system, and Fig. 2 shows the ***percent overshoot*** as well.

The system of Fig. 1 is an ***over-damped*** system with poles at $s=-2$ and $s=-5$. The ***settling times*** of the two poles are $4/2=2$ seconds and $4/5=0.8$ seconds. From the step response plot, it is clear the system settling time is ***close*** to that of the ***slower pole***.

The system of Fig. 2 is an ***under-damped*** system with ***natural frequency*** $\omega_n=\sqrt{25}=5\ \text{(rad/s)}$ and a ***damping ratio*** of $\zeta=5/2\omega_n=0.5$. The measured settling time of 1.62 seconds is consistent with the estimated settling time of $T_s=4/\zeta\omega_n=4/(5/2)=8/5\approx 1.6\ \text{(sec)}$.

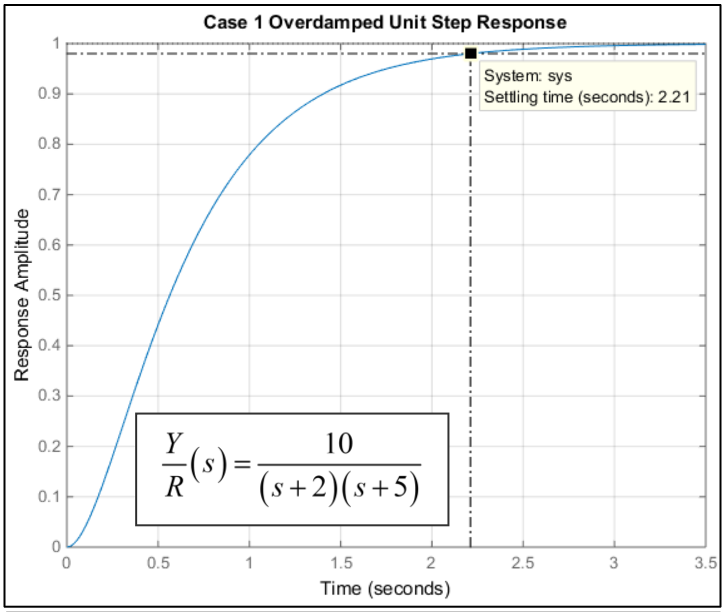

Fig. 1 Case 1 Second Order, ***Over-damped*** Unit Step Response

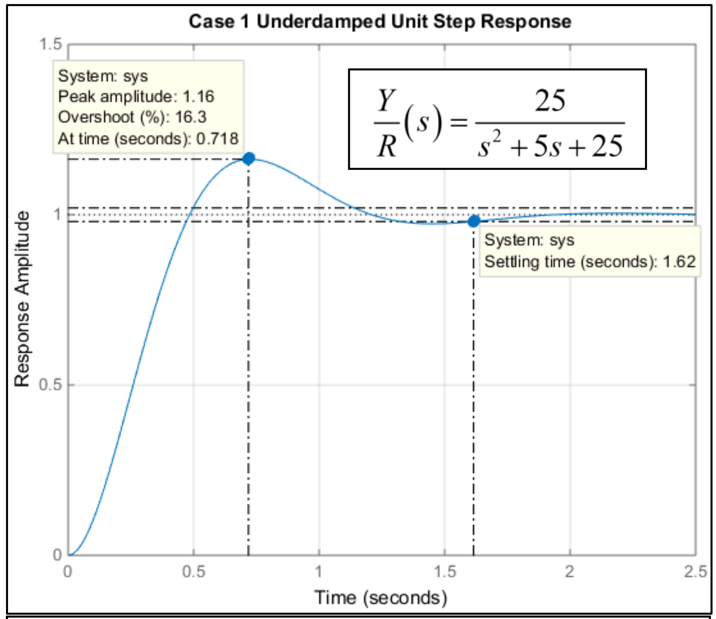

Fig. 2 Case 1 Second Order, ***Under-damped*** Unit Step Response

The measured percent overshoot of 16.3% is consistent with plot of percent overshoot versus damping ratio presented in earlier notes and Fig. 5.8 of the Dorf & Bishop text. The plot from previous notes is reproduced here for convenience of the reader. See Fig. 3 below.

Fig. 4 shows Case 1 second-order system unit step responses for ***various values*** of the ***damping ratio*** ζ . The natural frequency of the system is $\omega_n = 5$ (rad/s) . Note as the ***damping ratio*** gets ***smaller***, the responses become ***more oscillatory*** with ***larger overshoots***.

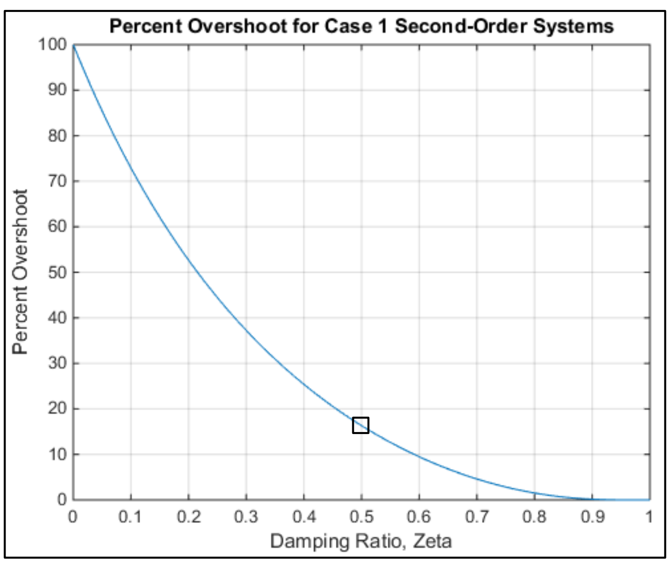

Fig. 3 Percent Overshoot vs. Damping Ratio ζ

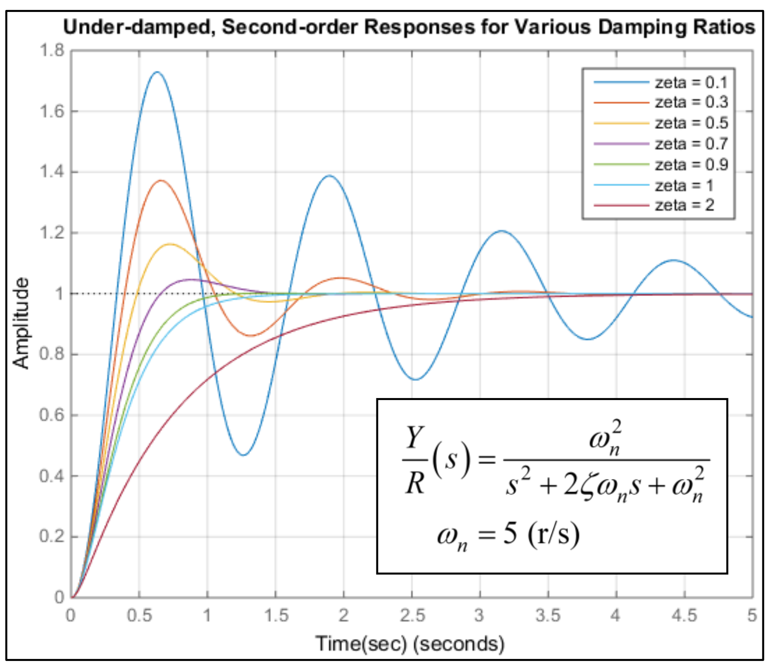

Fig. 4 Second Order, Case 1, Unit Step Responses

Case 2: $\frac{X}{R}(s)=\frac{(q/a)(s+a)}{s^2+ps+q}$

- The motion of Case 2 systems is complicated somewhat over that of Case 1 systems by the presence of the ***zero*** $(s+a)$ in the transfer function. The ***zero*** has ***little effect*** on the ***settling time*** of the system but ***can significantly affect the overshoot***.
- How ***much effect*** the zero has on overshoot depends on ***where*** it is (along the real axis) compared to the poles of the system.
  - ***Zero is far to the left of the poles***: it has ***little effect*** on the response of the system. This makes the response much like that of a Case 1 system.
  - ***Zero located near or inside the poles***: it will ***significantly affect*** the overshoot of the system.

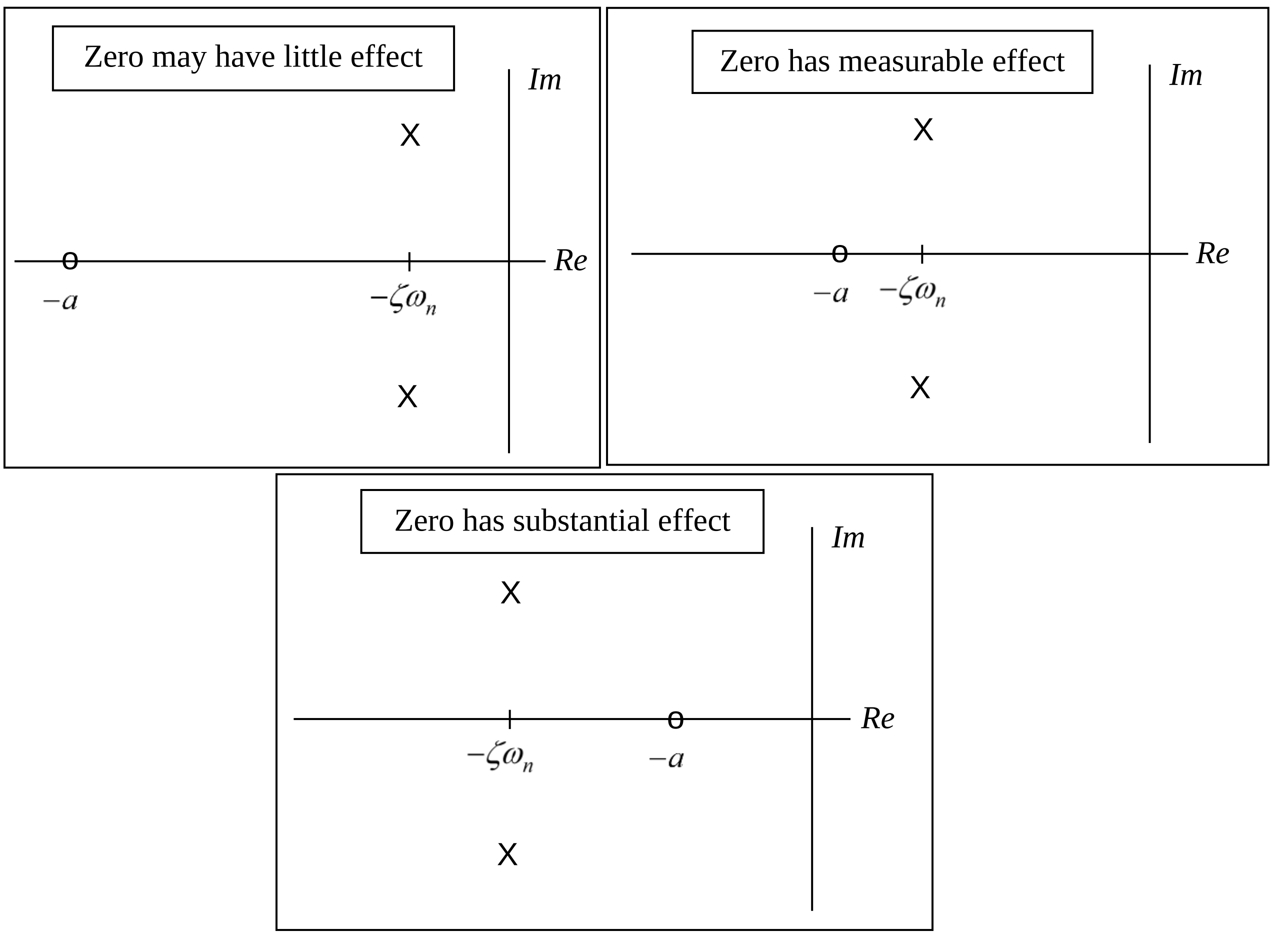

Fig. 4 below shows the step response of a Case 2 ***over-damped***, second order system. The system is the same as that shown in Fig. 1, except it has a zero at $s=-1$. Even though the system is over-damped, the presence of the zero ***causes significant overshoot***. The settling time of the system is unchanged.

Fig. 5 below shows the step response of a Case 2 ***under-damped***, second order system. The system is the same as that of Fig. 2, except it has a zero at $s=-1$. The zero ***increases the overshoot*** of the system from 16% to over

200%. The settling time has increased (somewhat) to 1.9 seconds. Note that the system narrowly escapes the 2% band at just about 1.6 seconds which is the settling time for the corresponding Case 1 system.

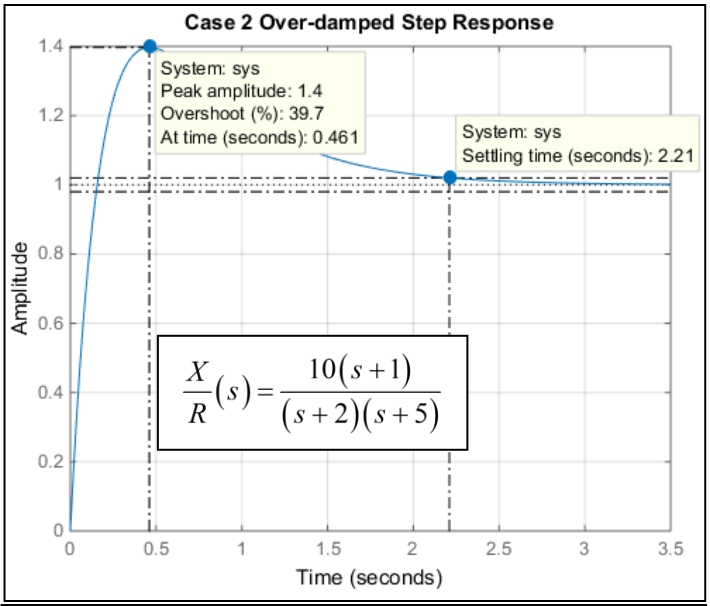

Fig. 4 Case 2 Second Order, Over-damped Step Response

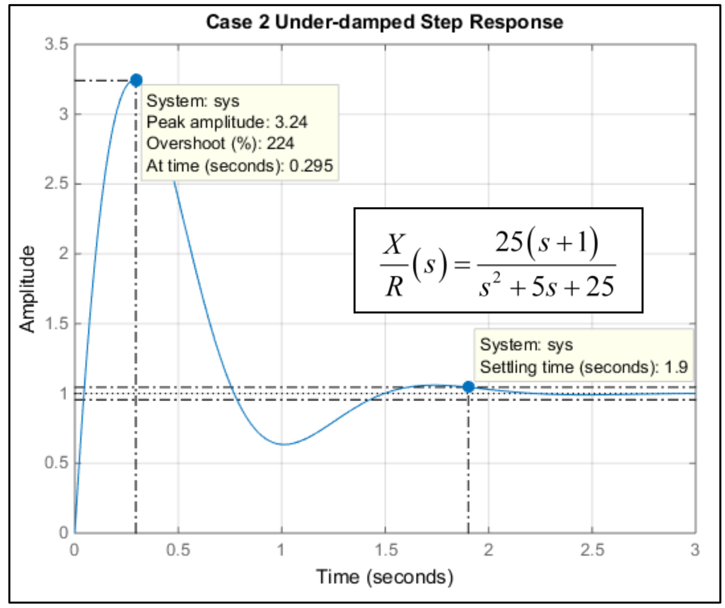

Fig. 5 Case 2 Second Order, Under-damped Step Response

The effects of the presence of a zero on the percent overshoot for under-damped and critically damped systems can be estimated using Fig. 6 below that was developed in previous notes. Parameter $\beta \triangleq a/\zeta\omega_n$ for the under-damped systems and $\beta \triangleq a/\alpha$ for the critically damped systems (repeated poles at $s=-\alpha$). This plot is similar to Fig. 5.13 in the Dorf and Bishop text.

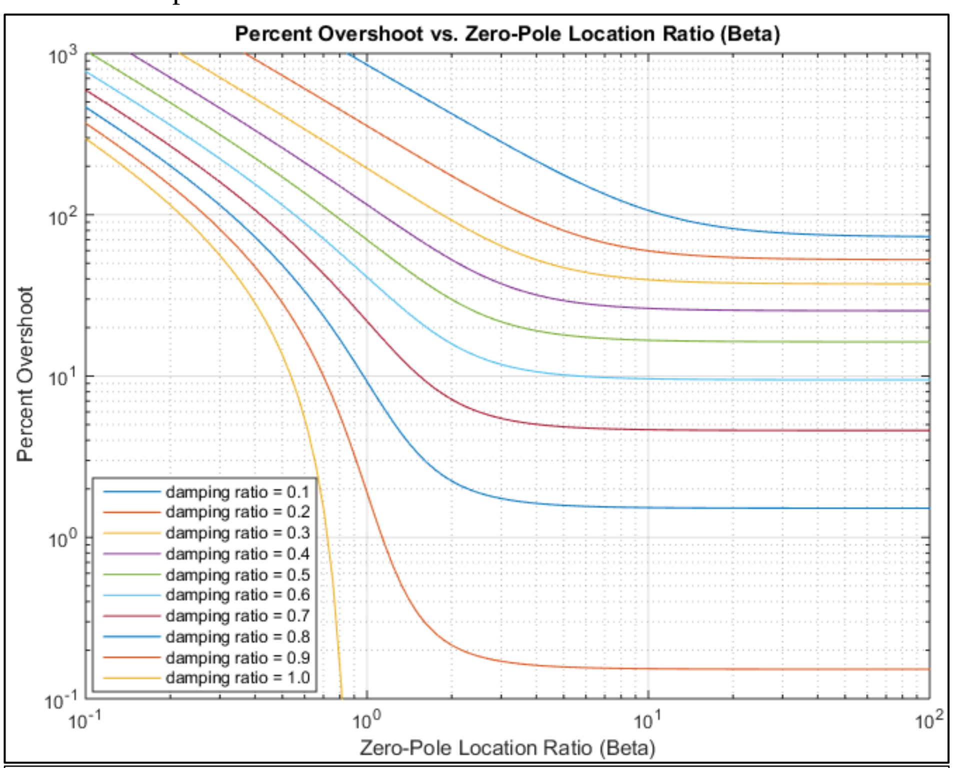

Fig. 6 Percent Overshoot vs. Zero-Pole Location Ratio β