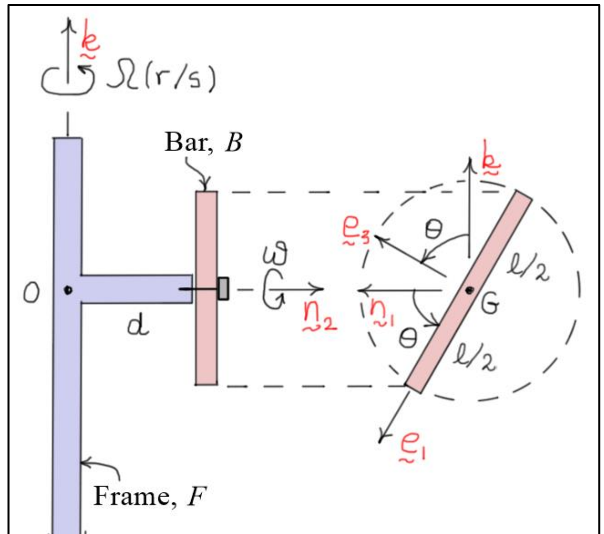
Intermediate Dynamics
Equations of Motion of Example System II

In previous notes for Example System II, ${}^R\underset{\sim}{\omega}_B$ the ***angular velocity*** of the bar, $[I_G]_{\underset{\sim}{e}}$ the ***inertia matrix*** (associated with $\underset{\approx}{I}_G$) resolved in the ***bar-fixed*** directions $B:(\underset{\sim}{e}_1,\underset{\sim}{n}_2,\underset{\sim}{e}_3)$, and $\underset{\sim}{H}_G$ the angular momentum of the bar about its mass-center were found to be



Reference Frames

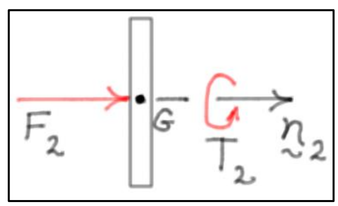
$$F:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{k})$$
$$B:(\underset{\sim}{e}_1,\underset{\sim}{n}_2,\underset{\sim}{e}_3)$$

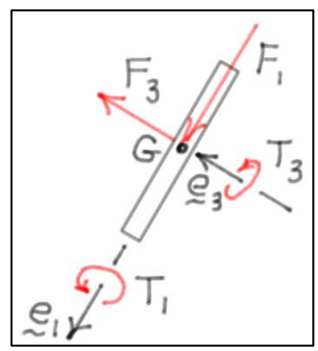
$$\boxed{{}^R\underset{\sim}{\omega}_B=(-\Omega S_\theta)\underset{\sim}{e}_1+\omega\underset{\sim}{n}_2+(\Omega C_\theta)\underset{\sim}{e}_3}$$

$$\boxed{[I_G]_{\underset{\sim}{e}}=\frac{m\ell^2}{12}\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}}$$

$$\boxed{\underset{\sim}{H}_G=\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\omega}_B=\frac{m\ell^2}{12}\left[\omega\underset{\sim}{n}_2+\Omega C_\theta\underset{\sim}{e}_3\right]}$$

The equations of motion of B can be found by applying the ***Newton/Euler equations*** to the free-body diagrams shown at the right.





$$\boxed{\begin{aligned}&\sum\underset{\sim}{F}=m\,{}^R\underset{\sim}{a}_G\\&\sum\underset{\sim}{M}_G=\left(\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\right)+\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\end{aligned}}$$

Given the angular rate $\Omega=\text{constant}$, the terms on the right side of the moment equation can be calculated as follows.

$${}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G=\frac{m\ell^2}{12}\begin{vmatrix}\underset{\sim}{e}_1&\underset{\sim}{n}_2&\underset{\sim}{e}_3\\-\Omega S_\theta&\omega&\Omega C_\theta\\0&\omega&\Omega C_\theta\end{vmatrix}\;\Rightarrow\;\boxed{{}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G=\frac{m\ell^2}{12}\left(\Omega^2 S_\theta C_\theta\underset{\sim}{n}_2-\omega\Omega S_\theta\underset{\sim}{e}_3\right)}$$

$$[I_G]_{\underset{\sim}{e}}\{\alpha\}_{\underset{\sim}{e}}=\frac{m\ell^2}{12}\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}-\omega\Omega C_\theta\\\dot{\omega}\\-\omega\Omega S_\theta\end{Bmatrix}\;\Rightarrow\;\boxed{\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B=\frac{m\ell^2}{12}\left(\dot{\omega}\underset{\sim}{n}_2-\omega\Omega S_\theta\underset{\sim}{e}_3\right)}$$

<u>Inverse Dynamics</u> (assuming Ω and ω are constant)

In this case, forces and torques are calculated to produce the desired motion.

Force Equations:

$$\boxed{\sum \underset{\sim}{F} = F_1 \underset{\sim}{e}_1 + F_2 \underset{\sim}{n}_2 + F_3 \underset{\sim}{e}_3 = m\, {}^R\underset{\sim}{a}_G = m\left(-d\,\Omega^2 \underset{\sim}{n}_2\right)} \quad \Rightarrow \boxed{\begin{cases} F_1 = F_3 = 0 \\ F_2 = -m\,d\,\Omega^2 \end{cases}} \text{ (inverse dynamics)}$$

Moment Equations:

$$\sum \underset{\sim}{M}_G = T_1 \underset{\sim}{e}_1 + T_2 \underset{\sim}{n}_2 + T_3 \underset{\sim}{e}_3 = \tfrac{1}{12} m\,\ell^2 \left[\left(\dot{\omega} + \Omega^2 S_\theta C_\theta\right)\underset{\sim}{n}_2 - 2\,\omega\Omega S_\theta \underset{\sim}{e}_3\right]$$

$$\Rightarrow \boxed{\begin{cases} T_1 = 0 \\ T_2 = \tfrac{1}{12} m\,\ell^2 \left(\dot{\omega} + \Omega^2 S_\theta C_\theta\right) \\ T_3 = -\tfrac{1}{6} m\,\ell^2\,\omega\Omega S_\theta \end{cases}} \text{ (inverse dynamics)}$$

Forward Dynamics of Bar B (assuming $\Omega = \text{constant}$, $\omega = \dot{\theta}$, and $\dot{\omega} = \ddot{\theta}$)

The same force and moment equations apply as written above. The difference here is that the angular motion of the bar is not constant. Consequently, the force components F_1, F_2, and F_3 and the torque components T_1 and T_3 are as calculated above. The ***moment equation*** about the $\underset{\sim}{n}_2$ direction, however, becomes a ***differential equation*** for ***tracking changes*** in θ. The torque component T_2 can be an ***applied torque*** or it can be a function of the angle θ and its derivatives as with spring and damping effects.

$$\boxed{\begin{array}{l} F_1 = F_3 = 0 \\ F_2 = -md\Omega^2 \end{array}} \qquad \boxed{\begin{array}{c} T_1 = 0 \\ \ddot{\theta} + \Omega^2 S_\theta C_\theta = 12 T_2 / m\ell^2 \\ T_3 = -\tfrac{1}{6} m\ell^2 \omega\Omega S_\theta \end{array}} \quad \text{(forward bar dynamics)}$$

Equilibrium Positions for the Bar

If torque $T_2(t) \equiv 0$, the bar exhibits ***equilibrium positions***. These positions can be calculated by setting all ***time-varying*** parts of the differential equation to ***zero***. That is,

$$\boxed{\Omega^2 S_\theta C_\theta = 0}$$

This equation is satisfied when $\boxed{\theta = 0, \pi/2}$. The stability of these steady-state positions determines how the bar responds when it is near them. These positions may be ***stable*** or ***unstable***. Generally, if stable, the bar will remain close to the equilibrium position if it is released near it. If the equilibrium position is unstable, the bar will move away from it when released.