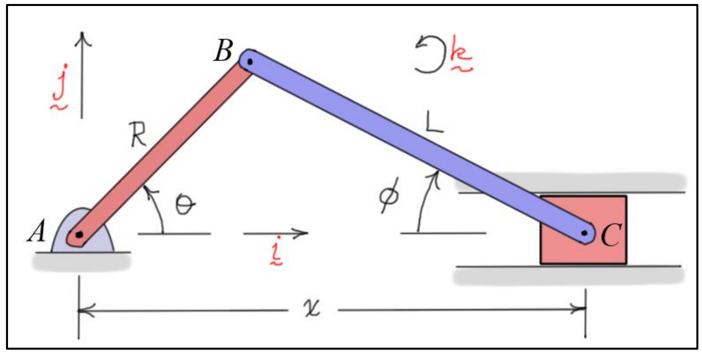
Intermediate Dynamics
Partial Velocities and the Slider Crank Mechanism

System Configuration

The figure shows a simple ***slider crank mechanism*** with no offset. Given the lengths of the links (R, L), the ***configuration*** of the system at any instant of time can be given by one or all of the ***generalized coordinates*** $(q_i) = (\theta, \phi, x)$. These coordinates are ***not independent***.



The following set of ***constraint equations*** can be used to relate the three coordinates.

$$\boxed{\begin{aligned} RS_\theta - LS_\phi &= 0 \\ RC_\theta + LC_\phi - x &= 0 \end{aligned}} \tag{1}$$

Here, S_θ and S_ϕ represent the sines of the angles, and C_θ and C_ϕ represent their cosines.

Partial Angular Velocities of the Links

Using the angles shown in the diagram, the ***angular velocities*** of the crank and connecting rod can be written as $\boxed{\underset{\sim}{\omega}_{AB} = \dot{\theta}\,\underset{\sim}{k}}$ and $\boxed{\underset{\sim}{\omega}_{BC} = -\dot{\phi}\,\underset{\sim}{k}}$. Using these expressions, the ***partial angular velocities*** of the links can be written as follows.

$$\boxed{\frac{\partial \underset{\sim}{\omega}_{AB}}{\partial \dot{\theta}} = \underset{\sim}{k}} \qquad \boxed{\frac{\partial \underset{\sim}{\omega}_{BC}}{\partial \dot{\phi}} = -\underset{\sim}{k}} \tag{2}$$

If the first of Eqs. (1) is differentiated, the angular rates $\dot{\theta}$ and $\dot{\phi}$ can be related as follows.

$$\boxed{R\dot{\theta}C_\theta = L\dot{\phi}C_\phi} \tag{3}$$

Using this result, another set of ***partial angular velocities*** can be defined as follows.

$$\boxed{\frac{\partial \underset{\sim}{\omega}_{AB}}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\dot{\theta}\,\underset{\sim}{k}\right) = \frac{\partial}{\partial \dot{\phi}}\left[\left(\frac{LC_\phi}{RC_\theta}\right)\dot{\phi}\,\underset{\sim}{k}\right] = \left(\frac{LC_\phi}{RC_\theta}\right)\underset{\sim}{k}} \tag{4}$$

$$\boxed{\frac{\partial \underset{\sim}{\omega}_{BC}}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(-\dot{\phi}\,\underset{\sim}{k}\right) = \frac{\partial}{\partial \dot{\theta}}\left[-\left(\frac{RC_\theta}{LC_\phi}\right)\dot{\theta}\underset{\sim}{k}\right] = -\left(\frac{RC_\theta}{LC_\phi}\right)\underset{\sim}{k}} \tag{5}$$

Eq. (4) indicates how ***changes*** in the ***angular velocity*** of the ***connecting rod*** *BC* ***affect*** the ***angular velocity*** of the ***crank*** *AB*, and Eq. (5) indicates how ***changes*** in the ***angular velocity*** of the ***crank*** *AB* ***affect*** the ***angular velocity*** of the ***connecting rod*** *BC*.

Partial Velocities of the Slider

The ***velocity*** of the slider can be written most simply as $\boxed{\underset{\sim}{v}_C = \dot{x}\,\underset{\sim}{i}}$. From this result, the following ***partial velocity*** can be defined.

$$\boxed{\frac{\partial \underset{\sim}{v}_C}{\partial \dot{x}} = \underset{\sim}{i}} \tag{6}$$

However, the ***velocity*** of the ***slider*** can also be written in terms of the crank or connecting bar angular rates using the ***relative velocity equation*** as follows.

$$\boxed{\underset{\sim}{v}_C = \underset{\sim}{v}_B + \underset{\sim}{v}_{C/B}} \tag{7}$$

Here,

$$\underset{\sim}{v}_B = \underset{\sim}{v}_{B/A} = \underset{\sim}{\omega}_{AB} \times \underset{\sim}{r}_{B/A} = \dot{\theta}\,\underset{\sim}{k} \times R\left(C_\theta \underset{\sim}{i} + S_\theta \underset{\sim}{j}\right) \quad \Rightarrow \boxed{\underset{\sim}{v}_B = R\dot{\theta}\left(-S_\theta \underset{\sim}{i} + C_\theta \underset{\sim}{j}\right)}$$

$$\underset{\sim}{v}_{C/B} = \underset{\sim}{\omega}_{BC} \times \underset{\sim}{r}_{C/B} = -\dot{\phi}\,\underset{\sim}{k} \times L\left(C_\phi \underset{\sim}{i} - S_\phi \underset{\sim}{j}\right) \quad \Rightarrow \boxed{\underset{\sim}{v}_{C/B} = -L\dot{\phi}\left(S_\phi \underset{\sim}{i} + C_\phi \underset{\sim}{j}\right)}$$

Substituting these results into Eq. (7) gives the following.

$$\boxed{\underset{\sim}{v}_C = \left(-R\dot{\theta}S_\theta - L\dot{\phi}S_\phi\right)\underset{\sim}{i} + \left(R\dot{\theta}C_\theta - L\dot{\phi}C_\phi\right)\underset{\sim}{j} = \left(-R\dot{\theta}S_\theta - L\dot{\phi}S_\phi\right)\underset{\sim}{i}} \tag{8}$$

Note the $\underset{\sim}{j}$ component is ***zero*** as a result of Eq. (3).

Using Eqs. (3) and (8), the following ***partial velocities*** can now be defined.

$$\underset{\sim}{v}_C = -\dot{\theta}\left[RS_\theta + \cancel{L}S_\phi\left(\frac{RC_\theta}{\cancel{L}C_\phi}\right)\right]\underset{\sim}{i} \quad \Rightarrow \boxed{\frac{\partial \underset{\sim}{v}_C}{\partial \dot{\theta}} = -R\left[S_\theta + \left(\frac{C_\theta S_\phi}{C_\phi}\right)\right]\underset{\sim}{i}} \tag{9}$$

$$\underset{\sim}{v}_C = -\dot{\phi}\left[\cancel{R}S_\theta\left(\frac{LC_\phi}{\cancel{R}C_\theta}\right) + LS_\phi\right]\underset{\sim}{i} \quad \Rightarrow \boxed{\frac{\partial \underset{\sim}{v}_C}{\partial \dot{\phi}} = -L\left[S_\phi + \left(\frac{C_\phi S_\theta}{C_\theta}\right)\right]\underset{\sim}{i}} \tag{10}$$