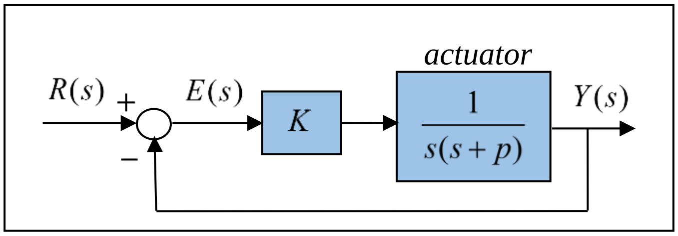
Introductory Control Systems
Closed-Loop Performance Examples

Proportional Control

The block diagram describing ***proportional control*** of a ***simple hydraulic actuator*** is shown below. The system has ***two parameters***, the proportional gain K and the system parameter p. The system parameter p represents how ***quickly*** the actuator gets to full speed.



Problem: Select the parameters K and p so the closed-loop system has:

a) As fast a response as possible with less than or equal to 5% overshoot

b) A settling time, $T_s \le 4$ (sec)

Solution:

1. The closed-loop transfer function is second order, $\boxed{\dfrac{Y(s)}{R(s)} = \dfrac{K}{s^2 + ps + K}}$. The gain K is seen here to affect the closed-loop system's stiffness, but not its damping. For as fast a response as possible with less than or equal to 5% overshoot, choose $\boxed{\zeta = 0.7}$. The system will respond more slowly if smaller overshoots are specified.

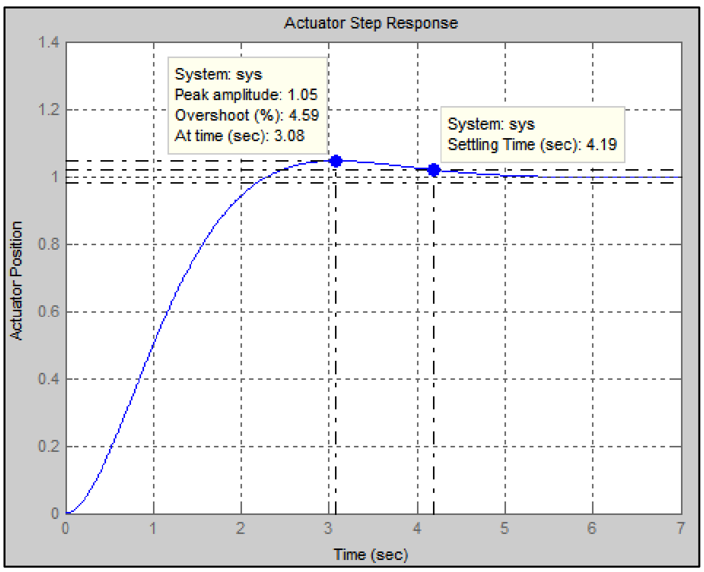
2. For a settling time, $T_s \le 4\ \text{(sec)}$, set $\boxed{T_s = \dfrac{4}{\zeta\omega_n} \le 4\ \text{(sec)}}$, or for the slowest response, set $\boxed{\zeta\omega_n = 1}$. With $\zeta = 0.7$, $\boxed{\omega_n = 1.4286\ \text{(rad/s)}}$.

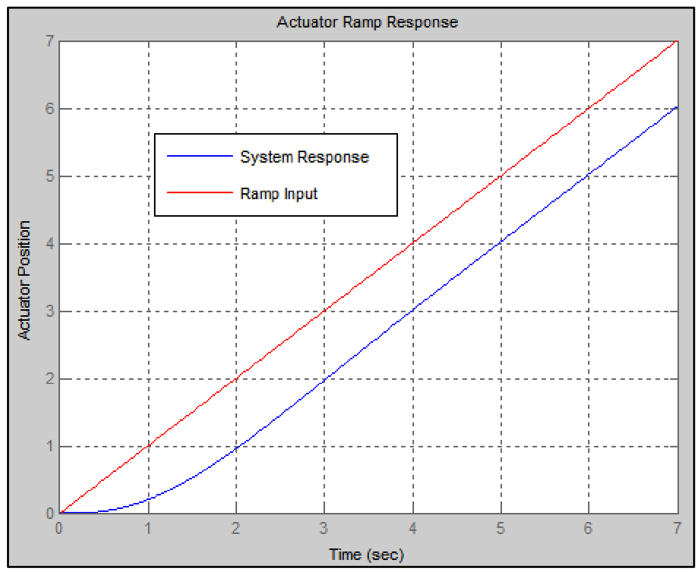
3. Hence, the system parameters are $\boxed{\begin{array}{l} p = 2\zeta\omega_n = 2 \\ K = \omega_n^2 = 2.04 \end{array}}$.

4. The ***error transfer function*** of this system is $\frac{E(s)}{R(s)} = \frac{s(s+p)}{s^2 + ps + K}$. Clearly, as a type 1 system, there is ***zero steady state error*** for a ***step input***, and the ***steady state error*** for a ***unit ramp input*** is $e_{ss} = \lim_{s \to 0} \left(s \cdot \frac{1}{s^2} \cdot \frac{E(s)}{R(s)} \right) = \frac{p}{K}$.

5. Note that since this is a type 1 second order system, this also represents ITAE optimal step response.

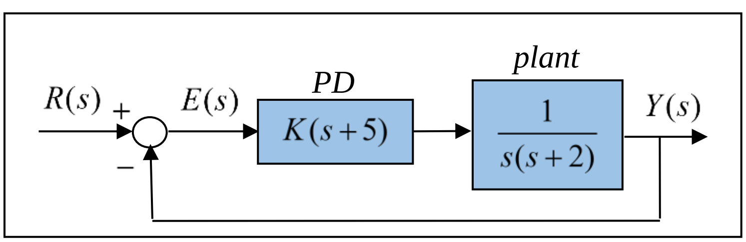
6. Step and Ramp responses:





<u>Proportional/Derivative Control</u>

The system from above is shown here with $p = 2$ and a *PD* controller with a zero at $s = -5$.



<u>Problem</u>: a) Find the values of K for which the closed-loop system has a damping ratio of $\zeta = 0.7$.

b) Find e_{ss} the steady state error of the system for a ramp input.

c) Plot the step and ramp responses of the system for the values of K found in (a).

Solution:

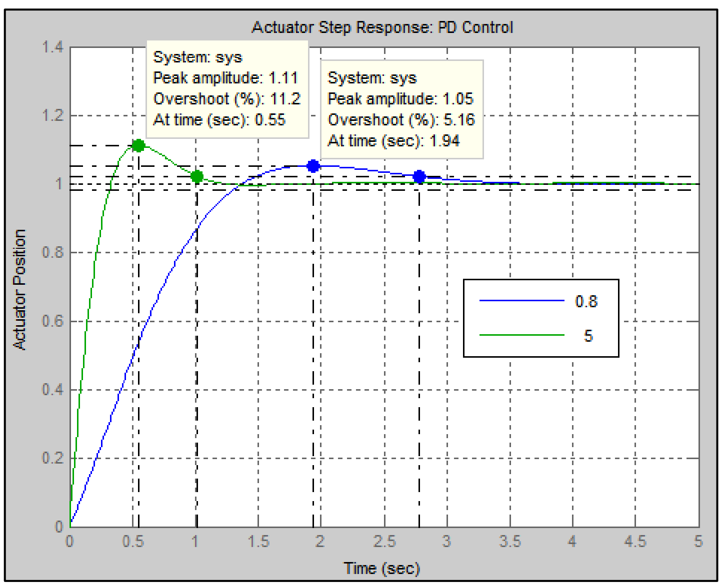
1. The closed-loop transfer function of this system is ***second order*** with a ***zero***. Note that the gain K now effects the ***stiffness*** and ***damping*** of the closed-loop system.

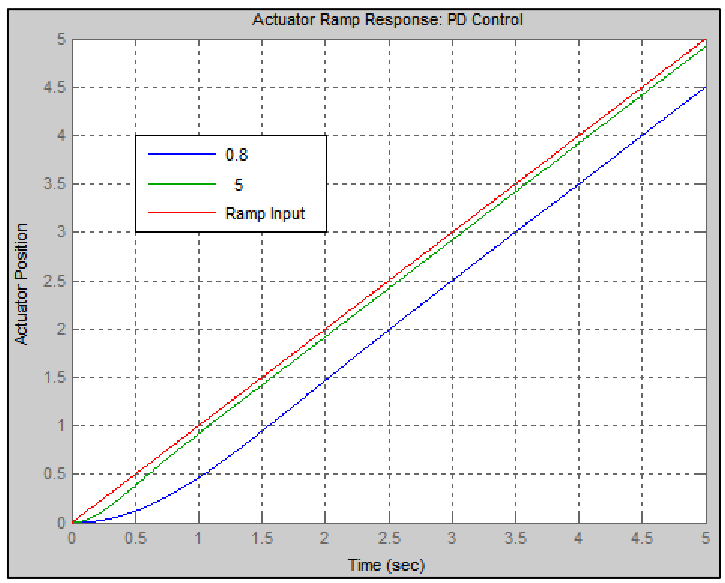
$$\boxed{\frac{Y(s)}{R(s)} = \frac{K(s+5)}{s^2+(K+2)s+5K}}$$

2. To find the values of K for which the closed loop system has a damping ratio of $\zeta = 0.7$, set $\boxed{2\zeta\omega_n = K+2}$ and $\boxed{\omega_n^2 = 5K}$. Solving these two equations simultaneously gives two solutions, 1) $K = 0.8$ and $\omega_n = 2$ (rad/s), and 2) $K = 5$ and $\omega_n = 5$ (rad/s).

3. The ***error transfer function*** of this system is $\boxed{\frac{E(s)}{R(s)} = \frac{s(s+2)}{s^2+(K+2)s+5K}}$. Clearly, as a type 1 system, there is ***zero steady state error*** for a ***step input***, and the ***steady state error*** for a ***unit ramp input*** is $\boxed{e_{ss} = \lim_{s \to 0}\left(s \cdot \frac{1}{s^2} \cdot \frac{E(s)}{R(s)} \right) = \frac{2}{5K}}$.

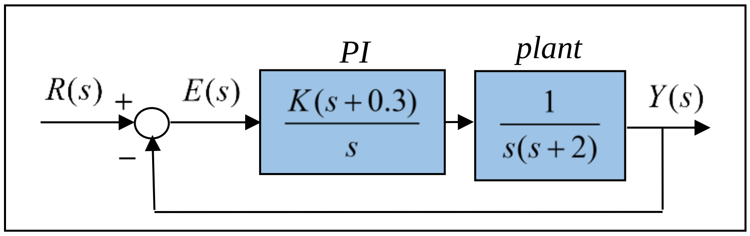
4. Step and Ramp responses:





Proportional/Integral Control

The system from above is shown here with $p = 2$ and a *PI* controller with a zero at $s = -0.3$.



Problem: a) Find the values of K for which the complex poles of the closed loop system have a damping ratio of $\zeta = 0.7$.
b) Find e_{ss} the steady state error of the system for a ramp input.
c) Plot the step and ramp responses of the system for the values of K found in (a).

Solution:

1. The closed loop transfer function of this system is ***third order*** with a ***zero***.

$$\boxed{\frac{Y(s)}{R(s)} = \frac{K(s+0.3)}{s^3 + 2s^2 + Ks + 0.3K}}$$

2. As a 3rd order system, it is more difficult to find the values of K for which the complex poles of the closed loop system have a damping ratio of $\zeta = 0.7$. After some ***trial and error***, the values of K are found to be $K = 1.27$ and $K = 1.9$. Complex poles have a damping ratio of $\zeta = 0.7$ when their real and imaginary parts are equal. (More about this later…)

3. The error transfer function of this system is $\boxed{\frac{E(s)}{R(s)} = \frac{s^2(s+2)}{s^3 + 2s^2 + Ks + 0.3K}}$. Clearly, as a type 2 system, there is ***zero steady state error for both step and ramp inputs***.

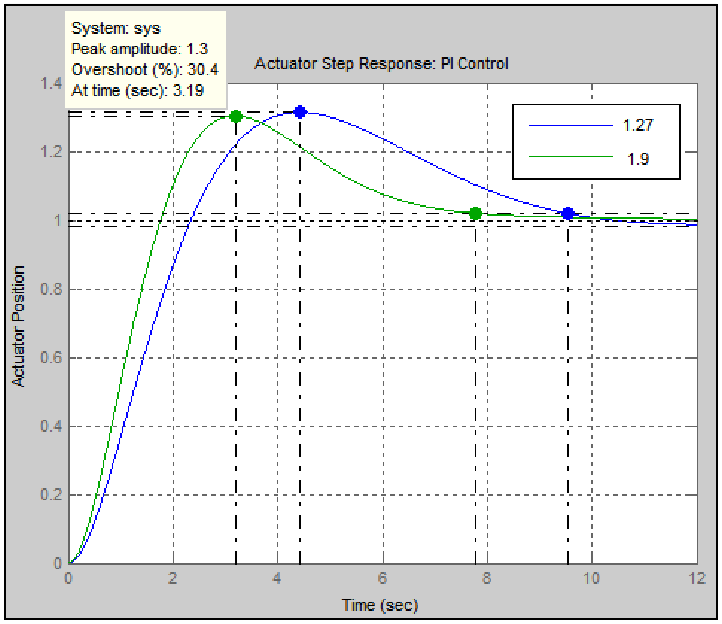
4. This transfer function has the ***form to be optimized for ramp input***. Using the ***location*** of the ***zero*** of the ***controller*** as the ***second parameter***, set

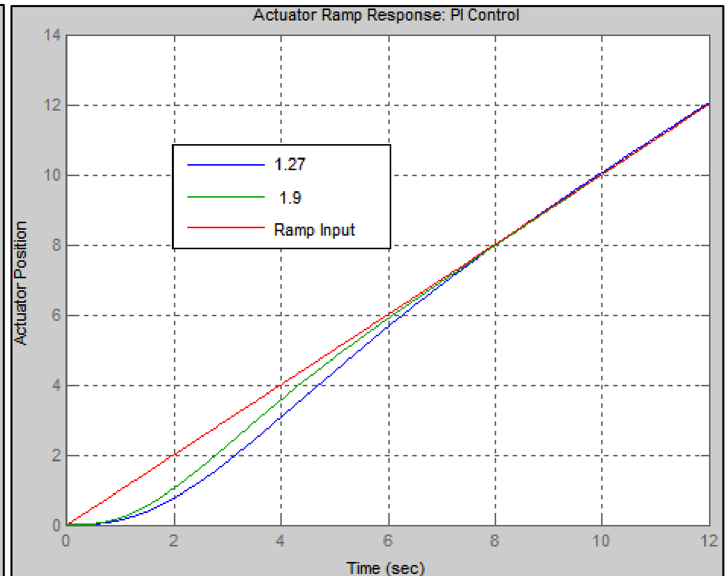
$$\boxed{s^3 + 2s^2 + Ks + Kz = s^3 + 1.75\omega_n s^2 + 3.25\omega_n^2 s + \omega_n^3}$$

Comparing coefficients, for optimal response, set $\boxed{K = 4.245}$ and $\boxed{z = 0.3374}$.

5. Step and Ramp responses:

The ***overshoot*** in the ***step response*** is ***amplified*** by the presence of the ***zero*** in the PI controller, but the ramp response looks good. See previous notes on the effects of a zero on second-order system response.





6. ITAE Optimal Ramp Response and Corresponding Step Response:

Note the ***improvement*** in the ***ramp response***. And again, the ***overshoot*** in the step response is ***amplified*** by the presence of the ***zero*** of the proportional-integral controller.