Intermediate Dynamics
Generalized Forces

General Definition

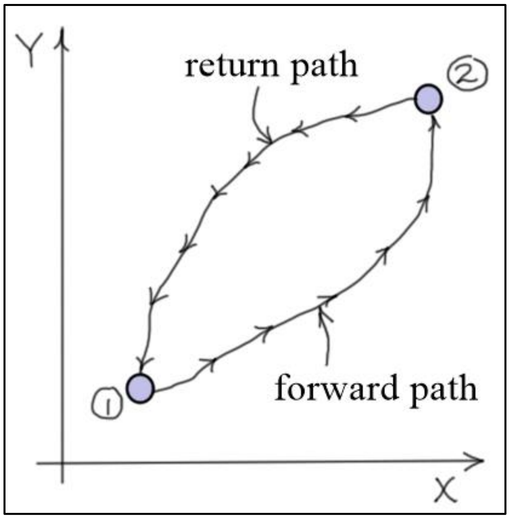
Given a mechanical system whose ***configuration*** is defined by a set of ***generalized coordinates*** $q_k \; (k = 1,\ldots,n)$, the ***generalized forces*** associated with the "n" ***generalized coordinates*** are defined as follows.

$$F_{q_k} = \sum_{\substack{forces\\(i)}} \left(\underset{\sim}{F}_i \cdot \frac{\partial\, {}^R\underset{\sim}{v}_i}{\partial \dot{q}_k} \right) + \sum_{\substack{torques\\(j)}} \left(\underset{\sim}{M}_j \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_j}{\partial \dot{q}_k} \right) \qquad (k = 1,\ldots,n) \tag{1}$$

Here the index "i" represents each of the forces, the index "j" represents each of the torques, and the index "k" represents each of the generalized coordinates. The forces $\underset{\sim}{F}_i$ and the torques $\underset{\sim}{M}_j$ that have ***non-zero*** contributions to this sum are called ***active*** forces and torques. The ***units*** of a ***generalized force*** are generally ***pounds*** or ***foot-pounds*** (or, equivalently, Newtons or Newton-meters) depending on the ***units*** of the ***generalized coordinates***.

Conservative and Nonconservative Forces

Consider a particle that moves from position 1 to position 2 along one path (forward path) and back again to position 1 along a second path (return path) as shown in the diagram. A force acting on the particle is said to be ***conservative*** if the ***net work*** it does over the closed path is ***zero***. Suppose, for example, that the work done by the force as the particle moves from position 1 to position 2 is ***positive***, then the force does the same amount of work as the particle returns to position 1, except that this work is ***negative***.



In this way, ***conservative forces do not permanently add or remove energy from a system***. When the conservative force is doing negative work, the system is said to be gaining ***potential energy*** that can later be transformed back into ***kinetic energy***. It is also true of conservative forces that the work done in moving from one position to another is ***independent of the path*** of the particle. Examples include weight forces and spring forces and torques.

Forces whose ***net work*** around a closed circuit are ***not zero*** are called ***nonconservative forces***. The work done by these forces as a particle moves from one position to another is ***dependent on the path*** of the particle. Examples include ***friction*** and ***damping*** forces.

<u>Conservative Forces and Potential Energy (V)</u>

The ***generalized force*** associated with ***conservative forces*** and/or ***torques*** can be written in terms of a ***potential energy function***, V. For weight forces and linear spring forces and torques, the potential energy functions are

$V = mgy$ (y is the height of the particle ***above*** some ***arbitrary datum***)

$V = \frac{1}{2}ke^2$ (e is the ***elongation*** or ***compression*** of the spring (units of length))

$V = \frac{1}{2}k\theta^2$ (θ is the ***elongation*** or ***compression*** of the spring (radians or degrees))

For systems with ***multiple conservative forces*** and ***torques***, the ***system potential energy*** is the ***sum*** of the potential energies of each of the forces and torques. That is, $V = \sum V_i$.

If conservative forces and/or torques do work on a system (i.e. if they are ***active***), their contribution to the ***generalized forces*** can be calculated using the ***general definition*** of Eq. (1), ***or*** they can be calculated using the ***potential energy function*** V as defined below in Eq. (2).

$$\boxed{F_{q_k} = -\left(\frac{\partial V}{\partial q_k}\right)} \tag{2}$$

Eq. (1) requires the use of ***vector expressions*** while Eq. (2) uses the ***scalar potential energy function***.

<u>Viscous Damping and Rayleigh's Dissipation Function (R)</u>

One type of ***nonconservative force*** or ***torque*** is associated with ***viscous damping***. One way of modeling this phenomenon is to ***assume*** that the forces or torques are ***proportional*** to the ***relative velocity*** or ***relative angular velocity*** of the ***end-points*** of the damping element.

Force: $\underset{\sim}{F} = -c\underset{\sim}{v}$ Torque: $\underset{\sim}{M} = -c\dot{\theta}\underset{\sim}{n}$

Here, the direction of the unit vector $\underset{\sim}{n}$ is defined by the right-hand rule. For these types of nonconservative forces and torques, ***Rayleigh's dissipation function*** is defined to be

Force: $R=\frac{1}{2}c\left(\dot{x}_2-\dot{x}_1\right)^2$ Torque: $R=\frac{1}{2}c\left(\dot{\theta}_2-\dot{\theta}_1\right)^2$

Here, $\left(\dot{x}_2-\dot{x}_1\right)$ represents the ***relative velocity*** of the end-points of the element and $\left(\dot{\theta}_2-\dot{\theta}_1\right)$ represents the ***relative angular velocity*** of the end-points element. For systems with ***multiple*** proportional damping elements, the ***system dissipation function*** is defined to be the sum of the dissipation functions of the individual dampers. That is, $R=\sum R_i$.

The contribution of these forces and/or torques to the ***generalized forces*** can be calculated using the ***general definition*** of Eq. (1), ***or*** they can be calculated using ***Rayleigh's dissipation function*** as given below in Eq. (3).

$$\boxed{F_{q_k}=-\left(\frac{\partial R}{\partial \dot{q}_k}\right)} \qquad (3)$$

Again, Eq. (1) requires the use of ***vector expressions*** while Eq. (3) uses the ***scalar dissipation function***. For a more detailed presentation of Rayleigh's dissipation function, see the following references.

1. L.A. Pars, *A Treatise on Analytical Dynamics*, Ox Bow Press, 1965.
2. L. Meirovitch, *Methods of Analytical Dynamics*, McGraw-Hill, 1970.
3. L. Meirovitch, *Principles and Techniques of Vibrations*, Prentice-Hall, 1997.
4. H. Baruh, *Analytical Dynamics*, McGraw-Hill, 1999.