Introductory Control Systems
Introduction to Root Locus Diagrams

The ***dynamic characteristics*** (transient response and stability) of a linear system are dependent on the ***location*** of the ***poles*** of the system. When a compensator (controller) is used to provide closed-loop control of a system, the ***poles*** of the ***closed-loop transfer function*** determine the system's dynamic characteristics. The ***location*** of these poles are ***functions*** of the compensator's parameters. The root locus method provides a means to ***track*** the locations of the poles of the closed-loop system as a function of a ***single parameter*** of the system or compensator.

A ***root locus*** is the ***path*** of a pole in the *s*-plane (complex plane) as a function of a single parameter. A ***root locus diagram*** provides the ***paths*** of ***all poles*** of the system as a function of that parameter. Using this method, an analyst can choose values of the parameter that will yield ***acceptable*** dynamic characteristics.

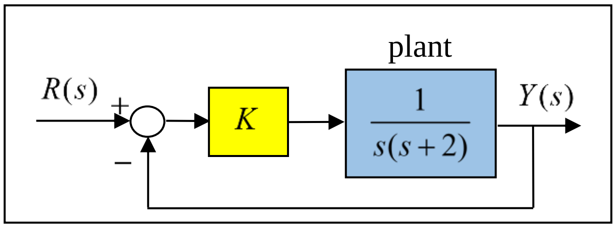
A ***root locus diagram*** can be ***generated*** from ***graphical considerations*** or by using ***computer software***. Clearly a design engineer will prefer to generate the details of these diagrams using a computer program but understanding of the graphical methods can provide the analyst with a better ***understanding*** of ***how*** the general characteristics of a root locus diagram will change as the type of compensator is altered. With this motivation, the notes that follow will discuss both approaches.

<u>Root Locus Diagrams – Basic Numerical Method</u>

As shown in the notes entitled "What is a Root Locus Diagram?", the most ***basic method*** for generating a root locus diagram is to find the ***characteristic equation*** of the system as a ***function*** of some parameter, say K, and then proceed to ***calculate the roots*** of the equation for a series of values of K. The roots are then plotted to form the root locus diagram. The ***movement*** of the poles are ***easily tracked***, but ***little information*** is available about how the diagram will change as the form of the controller is altered.

Root Locus Diagrams – Basic Graphical Method

To illustrate the ***basic graphical method***, consider the simple closed-loop system shown in the diagram. The closed-loop transfer function of the system is



$$\frac{Y}{R}(s)=\frac{G}{1+GH(s)}=\frac{\dfrac{K}{s(s+2)}}{1+\dfrac{K}{s(s+2)}}=\frac{K}{s(s+2)+K}$$

The ***characteristic equation*** of the system can then be written as

$$1+K\underbrace{\left(\frac{1}{s(s+2)}\right)}_{P(s)}=0 \quad \text{or} \quad s(s+2)+K=0 \qquad (1)$$

As will be discussed in subsequent notes, the first of these is considered the ***standard form*** for root locus diagrams. The polynomial ratio $P(s)$ is defined as the multiplier of the parameter K in the characteristic equation (Eq. (1)). Generally, both the numerator and denominator of $P(s)$ will be polynomials.

To generate the root locus diagram $(K \geq 0)$ for this system, first rewrite the second form of the characteristic equation as follows

$$s(s+2)=-K$$

Now replace "*s*" and "*s*+2" in this equation with the polar forms (recall that $j=\sqrt{-1}$)

$$s=a_1 e^{j\alpha_1} \quad \text{and} \quad s+2=a_2 e^{j\alpha_2}$$

Substituting these forms into the characteristic equation gives

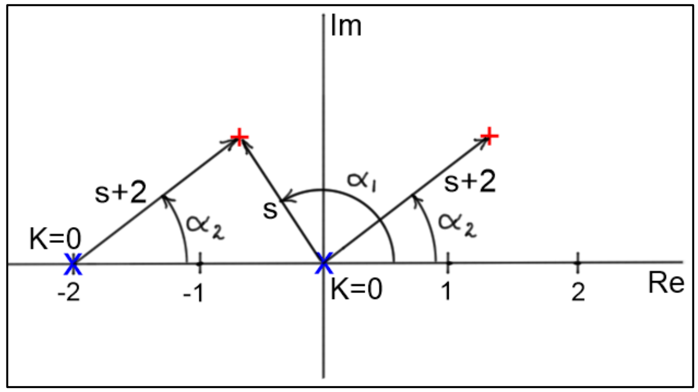
$$a_1\, a_2\, e^{j(\alpha_1+\alpha_2)}=-K+0\,j=K\,e^{jn\pi}$$

This equation provides the following two scalar equations

$$a_1 a_2 = K \quad \text{and} \quad \alpha_1+\alpha_2=n\pi \quad (n=\pm 1, \pm 2, \ldots) \qquad (2)$$

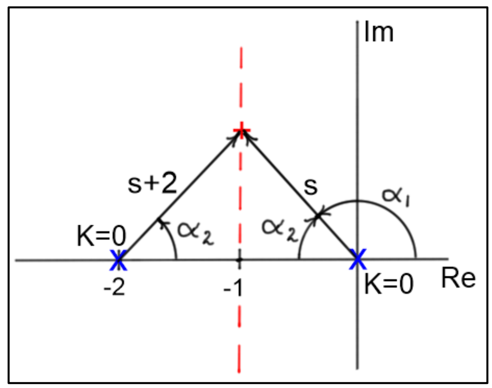
So, if a point in the *s*-plane is a root of the characteristic equation, it must satisfy these two equations. The paths of the poles start with $K = 0$ and the proceed as $K \to \infty$. Note from the characteristic equation that if $K = 0$, $s = 0$ and $s = -2$.

<u>Note on Geometry in the *s*-Plane</u>:

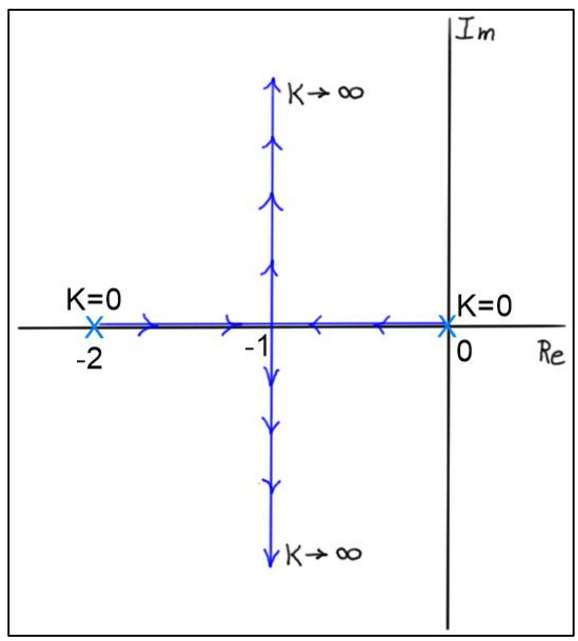


The diagram to the right shows a geometric representation of the terms "s" and "$s+2$" in the *s*-plane. The vectors are shown with both tails at the origin and with both ends at "s". When both ends are at "s", the tails start at the poles of $P(s)$ at $s = 0$ and $s = -2$.

<u>Locations on the Root Locus Diagram</u>



For a point to be on the root locus diagram of the example system shown above, the second of Eqs. (2) requires the sum of the angles α_1 and α_2 must be a multiple of π (or 180 degrees). The diagram to the right shows that all points on the dashed red line crossing the real axis at $s = -1$ satisfy this criterion.



At any point on the dashed red line that is off the real axis, the vectors for "s" and "$s+2$" form an isosceles triangle as shown. Both angles are non-zero, but their sum is π. Note also that any point on the real axis in the range $-2 < s < 0$ also satisfies the angle criterion. At these points $\alpha_1 = \pi$ and $\alpha_2 = 0$. Note finally that the value of K increases as the points move further away from the starting points at $s = 0$ and $s = -2$. The final root locus diagram has two branches and is shown in the diagram to the right.

Values of K on the Two Branches

The value of K at any point on the branches can be calculated using the first of Eqs. (2). The values in following table were found using this approach.

Location	s	$s+2$	a_1	a_2	K
$-0.5+0j$	$-0.5+0j$	$1.5+0j$	0.5	1.5	0.75
$-1.5+0j$	$-1.5+0j$	$0.5+0j$	1.5	0.5	0.75
$-1+0j$	$-1+0j$	$1+0j$	1	1	1
$-1+1j$	$-1+1j$	$1+1j$	$\sqrt{2}$	$\sqrt{2}$	2
$-1-1j$	$-1-1j$	$1-1j$	$\sqrt{2}$	$\sqrt{2}$	2

The table shows the system has ***two real***, ***unequal poles*** for $0<K<1$, ***two real***, ***equal poles*** for $K=1$, and a ***pair of complex conjugate poles*** for $K>1$. Hence, the closed-loop system is ***over-damped*** for $0<K<1$, ***critically damped*** for $K=1$, and ***under-damped*** for $K>1$.

Closure

Following the above procedure for ***more complex systems*** would be ***very tedious*** and ***time consuming***. Fortunately, there is a ***much faster***, ***less detailed approach*** that can be taken. That approach is ***presented*** and is ***applied*** to a few example systems in the notes that follow. Recall the ***motivation*** for sketching a root locus diagram is for the analyst to have a ***clearer understanding*** of the general impact various forms of control will have on a system. Once a candidate method is found, more ***detailed root locus diagrams*** can be generated using ***computer software***.