Introductory Motion and Control
***Z* Transforms and the Zero-order-hold (*ZOH*)**

The ***Laplace transform*** is used to find ***transfer functions*** for ***continuous systems***. The ***Z-transform*** is used to find ***discrete transfer functions*** for systems. The systems themselves may be ***continuous*** or ***discrete***.

Consider a discretized function based on sampled data

$$y_d(t) = \sum_{k=0}^{\infty} y(kT)\delta(t-kT) \qquad \text{where} \qquad \delta(t-kT) = \begin{cases} 0 & \text{for } t \neq kT \\ 1 & \text{for } t = kT \end{cases}$$

Taking Laplace transforms of the above equation gives

$$\boxed{\mathfrak{L}(y_d(t)) = \sum_{k=0}^{\infty} y(kT)\mathfrak{L}(\delta(t-kT)) = \sum_{k=0}^{\infty} y(kT)e^{-skT}}$$

where e^{-skT} represents a ***time delay*** of kT seconds. Defining the variable $z = e^{sT}$, the *Z*-transform of a sequence of numbers (e.g. sampled data) $y(kT)$ is defined as

$$\boxed{Z(y(kT)) = Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k}}$$

This is an ***infinite series*** in the variable z. The term $y(kT)z^{-k}$ indicates the value $y(kT)$ occurs at time $t = kT$, a ***delay*** of kT seconds from the ***start*** of the sequence.

Like the ***Laplace transform***, the ***Z transform*** is a ***linear*** transformation. ***One*** important ***property*** of the *Z* transform provides the means to find ***difference equations*** associated with a discrete transfer function. That is,

$$Z\big(y(k-1)\big) = z^{-1}Z\big(y(k)\big) = z^{-1}Y(z) \quad \Rightarrow \quad \boxed{Z^{-1}\big(z^{-1}Y(z)\big) = y(k-1)}$$

Here, the condensed notation $y(k) \triangleq y(kT)$ is used for simplicity.

A ***second*** useful ***property***, the ***discrete final value theorem*** states

$$\boxed{\lim_{k\to\infty}\big(y(k)\big) = \lim_{z\to 1}\big((1-z^{-1})Y(z)\big)}$$

The continuous and discrete final value theorems can be used to ***verify*** that the ***final value*** produced by a continuous transfer function and an equivalent discrete transfer function are the ***same***.

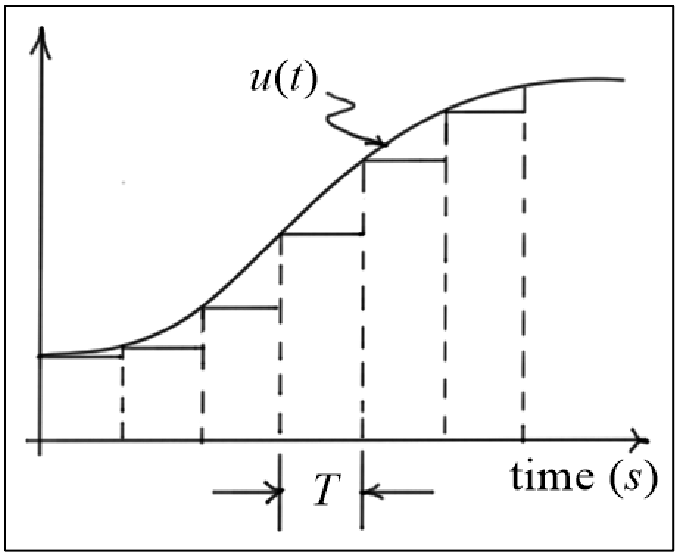
Digital-to-Analog Conversion and Zero-Order Hold (*ZOH*)

At each sampling time, the digital-to-analog converter (*DAC*) ***changes*** a digital value into an analog signal by ***sampling*** the data and ***holding*** the signal at that level until the next sample time occurs. This is referred to as a ***zero-order hold*** and is represented by the transfer function

$$G_{ZOH}(s) = \tfrac{1}{s} - \tfrac{1}{s}e^{-sT} = \frac{1-e^{-sT}}{s}$$

This transfer function represents the ***difference*** of ***two step functions***. The second step is delayed by T seconds (multiplication by e^{-sT}), so the difference is a ***square pulse*** of ***unit amplitude*** lasting for T seconds.

When applied to the sequence $u(kT)$ the output is a ***series*** of ***square pulses*** that represent $u(t)$. The ***smaller*** the sample period T, the ***better*** the characterization of $u(t)$.



Signal Representation Using a ***Zero-Order Hold***