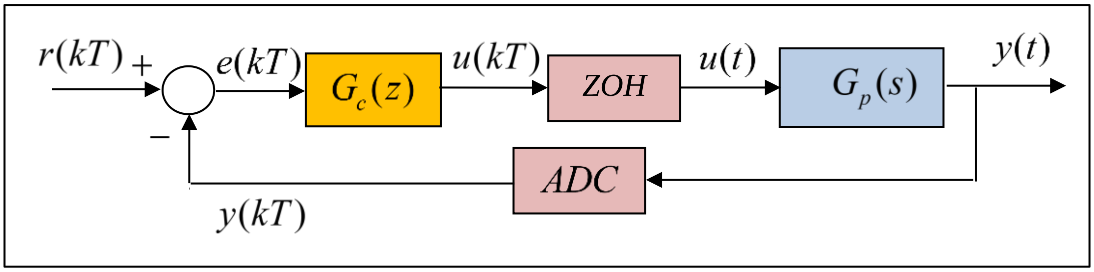
Introductory Motion and Control
Continuous and Equivalent Discrete Transfer Functions

The closed loop control system shown below has a ***digital compensator*** represented by the discrete transfer function $G_c(z)$ and a ***continuous plant*** represented by the continuous transfer function $G_p(s)$. The digital control signal $(u(kt))$ is shown to be converted into a continuous signal $u(t)$ using a ***zero-order hold*** (*ZOH*).



Block Diagram of a Continuous/Discrete Closed-Loop System

One common method of finding $G_c(z)$ involves a two-step process.

1. First, design its continuous counterpart $G_c(s)$ using continuous compensator design techniques (root locus and /or Bode diagrams)
2. Transform the resulting transfer function into a discrete form

Some authors refer to this as ***emulation***, because the ***discrete compensator*** emulates its ***continuous counterpart***. ***Tustin's approximation*** is one common method used to make this transformation. In this method, the ***substitution*** $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$ is made into the continuous transfer function to find $G_c(z)$.

For example, the continuous phase lead compensator

$$G_c(s) = \frac{3(s+5)}{s+15}$$

becomes

$$G_c(z) = \frac{3\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right)+5\right)}{\frac{2}{T}\left(\frac{z-1}{z+1}\right)+15} = \frac{3\left(\frac{2}{T}(z-1)+5(z+1)\right)}{\frac{2}{T}(z-1)+15(z+1)} = \frac{\left(15+\frac{6}{T}\right)z+\left(15-\frac{6}{T}\right)}{\left(15+\frac{2}{T}\right)z+\left(15-\frac{2}{T}\right)}$$

For a sample time $T = 0.001$ (sec), the final form of the compensator is

$$\boxed{G_c(z) = \frac{6015z - 5985}{2015z - 1985} = \frac{2.9851z - 2.97}{z - 0.9851}}$$

It can be shown that when Tustin's approximation is applied to an ***integral compensator***, the ***trapezoidal rule*** is used to approximate the integral.

MATLAB's "`c2d`" command can also be used to ***convert*** from ***continuous*** to ***discrete*** transfer functions. The methods of conversion include the ***Tustin*** approximation and a ***zero-order hold*** on the input to the transfer function.

```
>> num = 3*[1,5];
>> den = [1,15];
>> sys = tf(num,den)

   Transfer function:
     3 s + 15
     --------
     s + 15

>> sysD = c2d(sys,0.001,'tustin')

   Transfer function:
     2.985 z - 2.97
     --------------
     z - 0.9851

   Sampling time: 0.001
```

Given the ***discrete transfer function***, the compensator can be written as a ***difference equation*** as follows.

$$G_c(z) = \frac{U(z)}{E(z)} = \frac{6015z - 5985}{2015z - 1985} = \frac{2.985z - 2.97}{z - 0.9851}$$

or

$$(z - 0.9851)U(z) = (2.985z - 2.97)E(z)$$

Multiplying by z^{-1} and solving for $U(z)$ gives

$$(1 - 0.9851z^{-1})U(z) = (2.985 - 2.97z^{-1})E(z)$$

or

$$\boxed{U(z) = 0.9851z^{-1}U(z) + 2.985E(z) - 2.97z^{-1}E(z)}$$

The terms $U(z)$ and $E(z)$ refer to the compensator output and the error at the current time, and the terms $z^{-1}U(z)$ and $z^{-1}E(z)$ refer to the compensator output and the error at the previous time step. Hence, the above equation is ***equivalent*** to the ***difference equation***

$$\boxed{u(k) = 0.9851\,u(k-1) + 2.985\,e(k) - 2.97\,e(k-1)}$$

Another common method is the ***MPZ (matched pole-zero)*** Method. The *MPZ* method is based on ***mapping*** the ***poles*** and ***zeros*** of the continuous transfer function using the relationship $z = e^{sT}$ and ***preserving*** the low frequency gain. For example, consider a ***phase-lead*** or ***phase-lag*** type ***compensator*** of the form

$$\boxed{G_c(s) = K\left(\frac{s+a}{s+b}\right)}$$

In this case, the ***equivalent discrete*** transfer function for a given sample time *T* is

$$\boxed{G_c(z) = K'\left(\frac{z-e^{-aT}}{z-e^{-bT}}\right)}$$

Here, the constant K' is found by applying the ***final value theorems*** to each transfer function and equating the results.

$$\lim_{s\to 0}\left(\cancel{s}\cdot\tfrac{1}{\cancel{s}}\cdot G_c(s)\right) = K\left(\tfrac{a}{b}\right) = \lim_{z\to 1}\left(\left(\frac{1-z^{-1}}{1-z^{-1}}\right)\cdot G_c(z)\right) = K'\left(\frac{1-e^{-aT}}{1-e^{-bT}}\right)$$

or

$$\boxed{K' = K\left(\tfrac{a}{b}\right)\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)}$$

As with ***any numerical method*** (like Tustin's method described above), this method provides an ***approximation*** of the original continuous transfer function. The accuracy of the approximation is usually ***application dependent***. ***More details*** on this method may be found in Franklin, Powell, and Emami-Naeini, *Feedback Control of Dynamic Systems*, Prentice-Hall, 6th Ed. 2010.