Intermediate Dynamics
Linearization of Differential Equations of Motion

The differential equations of motion (EOM) derived using Newton's laws or Lagrange's equations may be ***linear*** or ***nonlinear***. If they are ***nonlinear***, it may be possible to ***linearize*** the equations about some ***equilibrium*** **(*steady state*)** ***positions***. From the resulting linear equations, the ***natural frequencies*** and ***mode shapes*** of the system for that position can be found. These provide the ***frequencies*** and describe the ***types*** of ***small deviations*** (motions) that the system has about that position. The equilibrium positions can be determined from Newton's law, the principle of virtual work, or directly from the nonlinear differential EOM by setting all time derivatives to zero.

Linearization of Functions of a Single Variable

A ***nonlinear function*** $y = f(x)$ can be expanded in a Taylor series around an equilibrium position, say x_{eq}. The Taylor series is an infinite series but can be truncated to find a linear approximation to $f(x)$ at the equilibrium position.

$$f\left(x_{eq} + \Delta x\right) = f\left(x_{eq}\right) + \Delta x\left[\frac{df}{dx}\right]_{x=x_{eq}} + \frac{(\Delta x)^2}{2}\left[\frac{d^2 f}{dx^2}\right]_{x=x_{eq}} + \cdots$$
$$\approx f\left(x_{eq}\right) + \Delta x\left[\frac{df}{dx}\right]_{x=x_{eq}}$$

Here, Δx represents an ***excursion*** from the ***equilibrium position***. If the excursions are ***small***, the approximation as stated in the second equation can be used. In this latter case, write

$$\boxed{\Delta f\left(x\right) = f\left(x_{eq} + \Delta x\right) - f\left(x_{eq}\right) = m\,\Delta x}$$

Here,

$$\boxed{m \triangleq \left.\frac{df}{dx}\right|_{x=x_{eq}}}$$

This is a ***linear*** relationship between ***changes*** in *f* and ***changes*** in *x*.

Linearization of Functions of Many Variables

A ***nonlinear function*** of multiple variables $y = f(x_1, x_2, \ldots, x_n) = f(\underset{\sim}{x})$ can also be expanded in a ***Taylor series*** around an ***equilibrium position***, say $\underset{\sim}{x}_{eq} = \left\{(x_1)_{eq}, (x_2)_{eq}, \ldots, (x_n)_{eq}\right\}$. As with functions of a single variable, the series can be truncated to find an approximation to $f(\underset{\sim}{x})$ at the equilibrium (steady-state) position.

$$f\left(\underset{\sim}{x}_{eq} + \Delta\underset{\sim}{x}\right) = f\left(\underset{\sim}{x}_{eq}\right) + \sum_{i=1}^{n} \Delta x_i \left[\frac{\partial f}{\partial x_i}\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{eq}} + \cdots$$

$$\approx f\left(\underset{\sim}{x}_{eq}\right) + \sum_{i=1}^{n} \Delta x_i \left[\frac{\partial f}{\partial x_i}\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{eq}}$$

Here, the vector $\Delta\underset{\sim}{x} = \left(\Delta x_1, \Delta x_2, \ldots, \Delta x_n\right)$ represents an ***excursion*** from the equilibrium position. As before, if the excursions are ***small***, the approximation as stated in the second equation can be used. In this latter case, write

$$\boxed{\Delta f\left(\underset{\sim}{x}\right) = f\left(\underset{\sim}{x}_{eq} + \Delta\underset{\sim}{x}\right) - f\left(\underset{\sim}{x}_{eq}\right) = \sum_{i=1}^{n} m_i \Delta x_i}$$

Here,

$$\boxed{m_i \triangleq \left.\frac{\partial f}{\partial x_i}\right|_{\underset{\sim}{x}=\underset{\sim}{x}_{eq}}}$$

This is a ***linear*** relationship between ***changes*** in *f* and ***changes*** in the ***elements*** of the state vector $\underset{\sim}{x}$.