

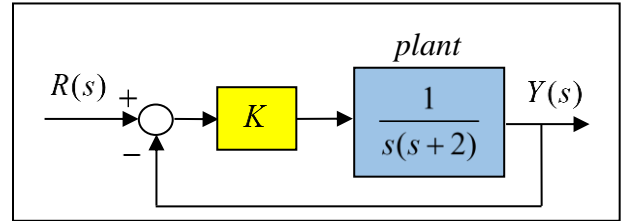
# Introductory Control Systems

## Root Locus (RL) Diagram Examples

1. Sketch the root locus diagram for the parameter  $K$  for the closed loop system shown in the diagram.

1) Characteristic Equation:

$$1 + GH(s) = 1 + K \underbrace{\left[ \frac{1}{s(s+2)} \right]}_{P(s)} = 0$$



2) Zeros of  $P(s)$ : none ( $n_z = 0$ )

Poles of  $P(s)$ :  $s = 0, -2$  ( $n_p = 2$ )  $\Rightarrow$  Number of branches = 2

Number of asymptotes:  $n_A = n_p - n_z = 2$

3) Poles on the real axis:  $-2 \leq s \leq 0$

Pole at  $s = 0$  moves to the **left** and pole at  $s = -2$  moves to the **right**, so there must be a break-away point in the range  $-2 \leq s \leq 0$ .

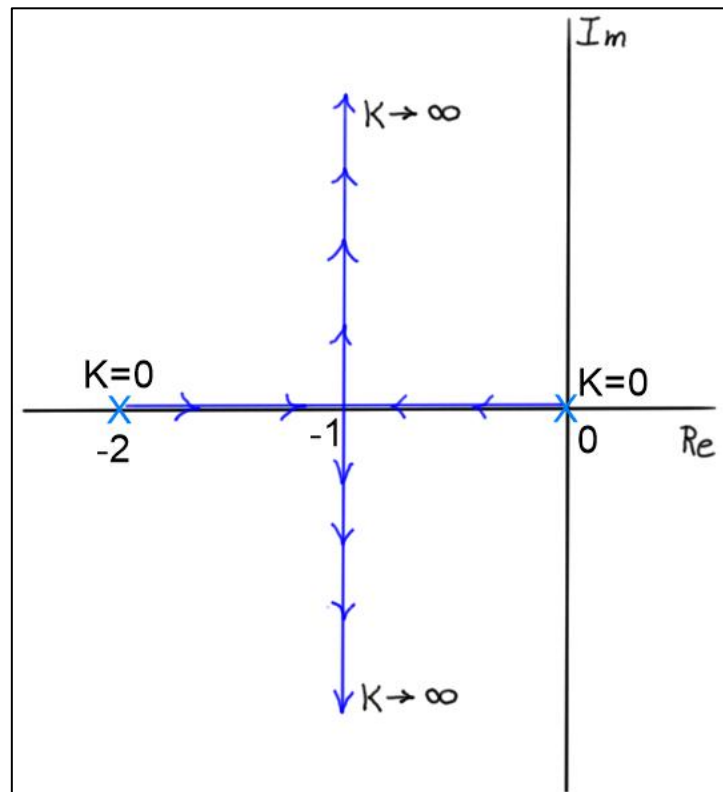
4) Asymptotes:  $\phi_A = \left( \frac{2m+1}{n_A} \right) 180 = \begin{cases} 90 \text{ (deg)} & (m=0) \\ 270 \text{ (deg)} & (m=1) \end{cases}$ ;  $\sigma_A = \frac{0-2}{2} = -1$

5) Break Points:

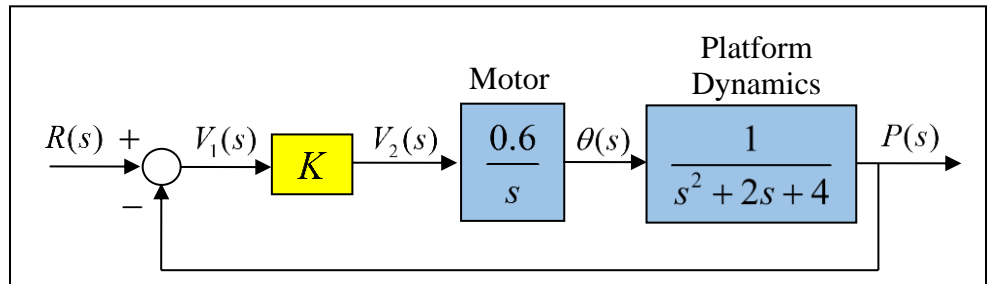
$$\frac{dP}{ds} = \frac{-1(2s+2)}{s^2(s+2)^2} = 0$$

$$\Rightarrow 2s+2=0$$

$$\Rightarrow s = -1$$



2. Sketch the root locus diagram for the parameter  $K$  for the closed loop system shown.



1) Characteristic Equation:  $1 + GH(s) = 1 + K \underbrace{\left[ \frac{0.6}{s(s^2 + 2s + 4)} \right]}_{P(s)} = 0$

2) Zeros of  $P(s)$ : none ( $n_z = 0$ )

Poles of  $P(s)$ :  $s = 0, -1 \pm 1.732j$  ( $n_p = 3$ )  $\Rightarrow$  Number of branches = 3

Number of asymptotes:  $n_A = n_p - n_z = 3$

3) Poles on the real axis:  $-\infty < s \leq 0$ . Pole at  $s = 0$  moves to the **left**; poles move to infinity along the **negative real axis** (which is one of the asymptotes).

4) Asymptotes:  $\phi_A = \left( \frac{2m+1}{n_A} \right) 180 = 60, 180, 300$  (deg),  $\sigma_A = \frac{0-1-1}{3} = -2/3$

Asymptotes at 60 and 300 deg. are shown in red.

5) Break Points: (break points must be on the real axis)

$$\frac{dP}{ds} = \frac{0.6(3s^2 + 4s + 4)}{s^2(s^2 + 2s + 4)^2} = 0 \Rightarrow \boxed{3s^2 + 4s + 4 = 0}$$

$$\Rightarrow \boxed{s = -0.667 \pm 0.94j} \Rightarrow \text{no break points}$$

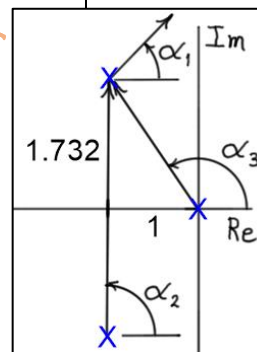
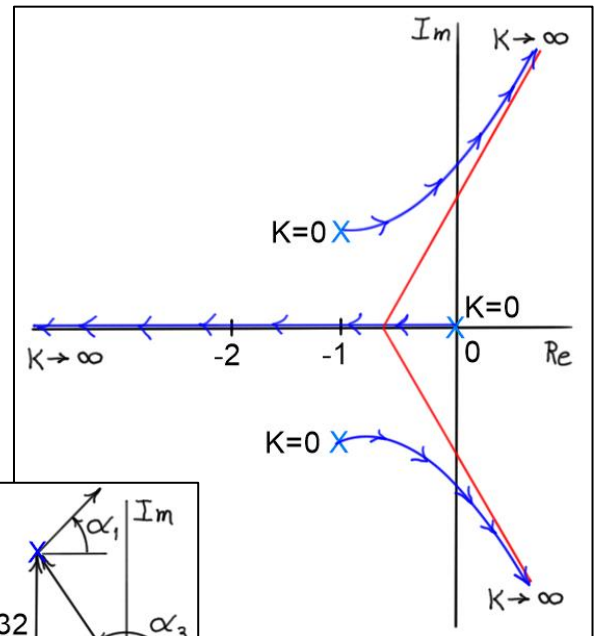
6) Exit angle from the upper complex pole:

$$\alpha_1 + \alpha_2 + \alpha_3 = 180$$

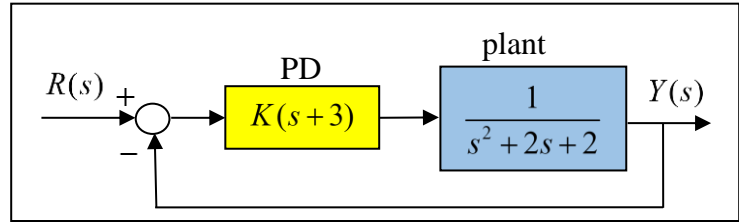
$$\alpha_1 + 90 + (180 - \tan^{-1}(1.732)) = 180$$

$$\Rightarrow \boxed{\alpha_1 = -30 \text{ (deg)}}$$

Exit angle from the lower complex pole is +30 (deg).



3. Sketch the root locus diagram for the parameter  $K$  for the closed loop system shown in the diagram.



1) Characteristic Equation:  $1 + GH(s) = 1 + K \underbrace{\frac{s+3}{s^2+2s+2}}_{P(s)} = 0$

2) Zeros of  $P(s)$ :  $s = -3$  ( $n_z = 1$ )

Poles of  $P(s)$ :  $s = -1 \pm 1j$  ( $n_p = 2$ )  $\Rightarrow$  Number of branches = 2

Number of asymptotes:  $n_A = n_p - n_z = 1$

3) Poles on the real axis:  $-\infty < s \leq -3$ . Poles **moving towards zero** at  $s = -3$ , and poles moving **towards  $\infty$  along the negative real axis** which is the only asymptote.

4) Asymptotes: No need for calculations. Lone asymptote is at 180 degrees.

5) Break Points:  $\frac{dP}{ds} = \frac{(1)(s^2+2s+2) - (s+3)(2s+2)}{(s^2+2s+2)^2} = 0 \Rightarrow \boxed{-s^2 - 6s - 4 = 0}$

$\Rightarrow \boxed{s = -0.764; -5.24}$ . The break point must be in the range  $-\infty < s \leq -3$ , so the break point for RL diagram is at  $s = -5.24$ .

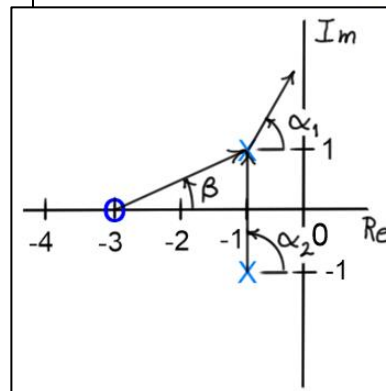
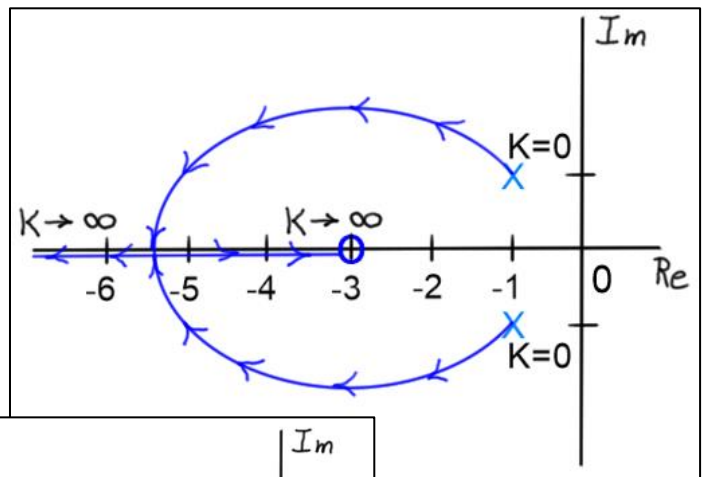
6) Exit angle from the upper complex pole:

$$\alpha_1 + \alpha_2 - \beta = 180$$

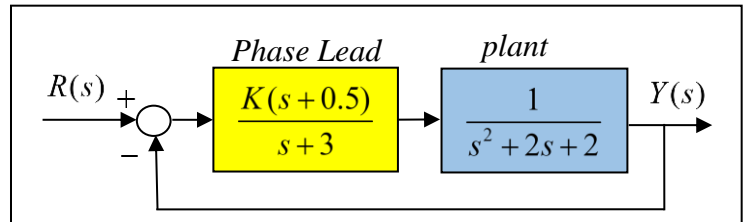
$$\alpha_1 + 90 - \tan^{-1}(1/2) = 180$$

$$\Rightarrow \boxed{\alpha_1 = 116.6 \text{ (deg)}}$$

Exit angle from the lower complex pole is  $-116.6$  (deg).



4. Sketch the root locus diagram for the parameter  $K$  for the closed loop system shown in the diagram.



1) Characteristic Equation:  $1 + GH(s) = 1 + K \underbrace{\left[ \frac{s + 0.5}{(s + 3)(s^2 + 2s + 2)} \right]}_{P(s)} = 0$

2) Zeros of  $P(s)$ :  $s = -0.5$  ( $n_z = 1$ )

Poles of  $P(s)$ :  $s = -3, -1 \pm 1j$  ( $n_p = 3$ )  $\Rightarrow$  Number of branches = 3

Number of asymptotes:  $n_A = n_p - n_z = 2$

3) Poles on the real axis:  $-3 \leq s \leq -0.5$

Poles moving away from pole at  $s = -3$ , and poles moving towards the zero at  $s = -0.5$ .

4) Asymptotes:

$$\phi_A = \left( \frac{2m+1}{n_A} \right) 180 = \begin{cases} 90 \text{ (deg)} & (m=0) \\ 270 \text{ (deg)} & (m=1) \end{cases}, \quad \sigma_A = \frac{-3 + 2(-1) - (-0.5)}{n_A} = \frac{-4.5}{2} = -2.25$$

5) Break Points:

$$\frac{dP}{ds} = \frac{(1)(s+3)(s^2+2s+2) - (s+0.5)(3s^2+10s+8)}{(s+3)^2(s^2+2s+2)^2} = \frac{2s^3 + 6.5s^2 + 5s - 2}{(s+3)^2(s^2+2s+2)^2} = 0$$

$$\Rightarrow 2s^3 + 6.5s^2 + 5s - 2 = 0$$

$$s = 0.285; -1.77 \pm 0.62j$$

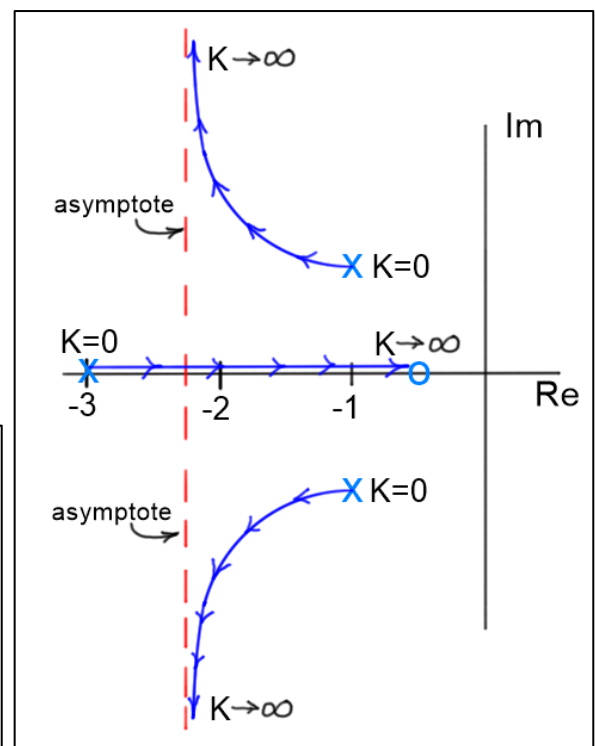
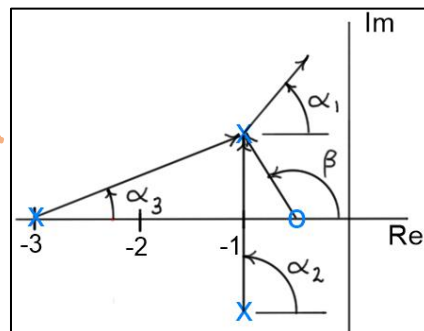
Break points on the RL diagram must be in the range  $-3 \leq s \leq -0.5$ , so there are **no break points**.

6) Exit angle from the upper complex pole:

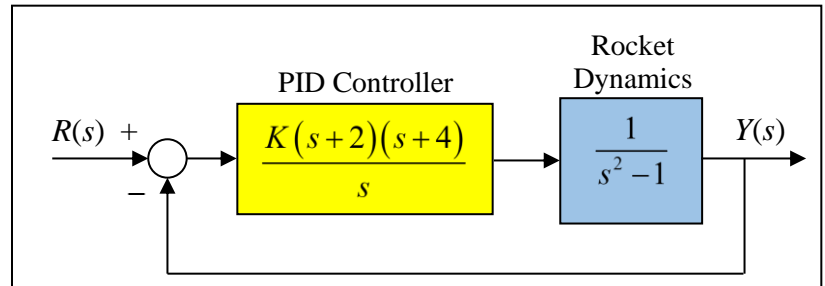
$$\alpha_1 + \alpha_2 + \alpha_3 - \beta = 180$$

$$\alpha_1 + 90 + \tan^{-1}(1/2) - (180 - \tan^{-1}(1/0.5)) = 180$$

$$\Rightarrow \alpha_1 = 180 \text{ (deg)}$$



5. Sketch the root locus diagram for the parameter  $K$  for the closed loop system shown in the diagram.



1) Characteristic Equation:  $1 + GH(s) = 1 + K \underbrace{\frac{(s+2)(s+4)}{s(s^2-1)}}_{P(s)} = 0$

2) Zeros of  $P(s)$ :  $s = -2, -4$  ( $n_z = 2$ )

Poles of  $P(s)$ :  $s = 0, \pm 1$  ( $n_p = 3$ )  $\Rightarrow$  Number of branches = 3

Number of asymptotes:  $n_A = n_p - n_z = 1$

3) Poles on the real axis (3 segments):  $-\infty < s < -4$ ,  $-2 < s < -1$ , and  $0 < s < +1$

Poles move to the **right** from  $s = 0$ ; poles move to the **left** from  $s = +1$ ; poles move **left** from  $s = -1$ ; poles move **towards** the zero at  $s = -2$ ; poles move **towards** the zero at  $s = -4$ ; poles move to infinity along the **negative real axis** which is the only asymptote.

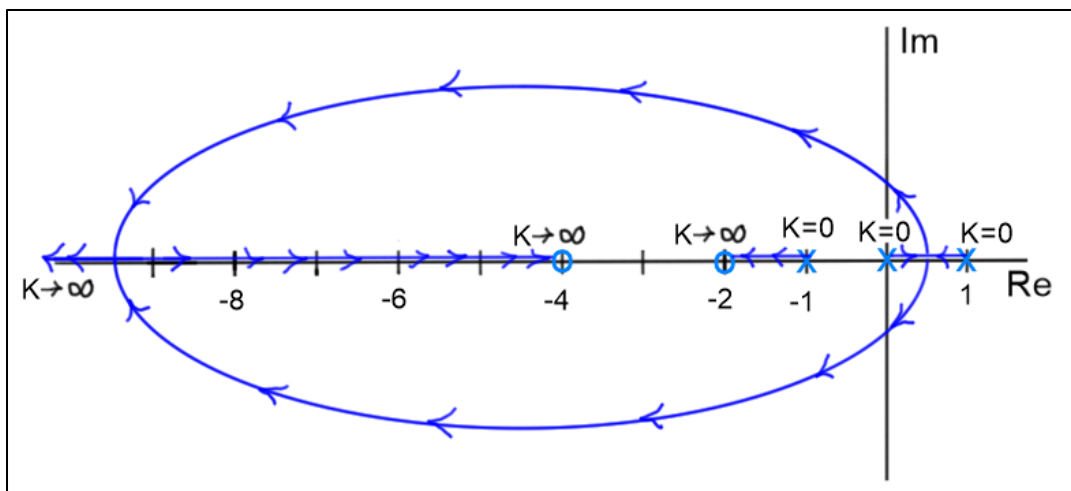
4) Asymptote:  $\phi_A = 180$  (deg)

5) Break Points:

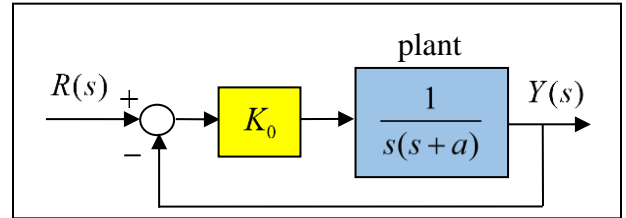
$$\frac{dP}{ds} = \frac{d}{ds} \left( \frac{(s+2)(s+4)}{s(s^2-1)} \right) = \frac{(2s+6)(s^3-s) - (s^2+6s+8)(3s^2-1)}{s^2(s^2-1)^2} = 0$$

$$\Rightarrow s^4 + 12s^3 + 25s^2 - 8 = 0 \Rightarrow s = -9.33, -2.495, -0.68, 0.5054$$

$\Rightarrow$  break points on RL diagram at  $s = -9.33, 0.5054$



6. Sketch the root locus diagram for the parameter  $a$  for the closed loop system shown in the diagram.



1) Characteristic Equation:  $1 + GH(s) = 1 + \left[ \frac{K_0}{s(s+a)} \right] = 0 \Rightarrow 1 + a \underbrace{\left[ \frac{s}{s^2 + K_0} \right]}_{P(s)} = 0$

In this case, some *algebraic manipulation* is required to put the characteristic equation in standard form.

2) Zeros of  $P(s)$ :  $s = 0$  ( $n_z = 1$ )

Poles of  $P(s)$ :  $s = \pm\sqrt{K_0} j$  ( $n_p = 2$ )  $\Rightarrow$  Number of branches = 2

Number of asymptotes:  $n_A = n_p - n_z = 1$

3) Poles on the real axis:  $-\infty < s < 0$

Poles move *towards* the zero at  $s = 0$ , and poles move to infinity along the *negative real axis* which is the only asymptote.

4) Asymptotes:  $\phi_A = 180$  (deg)

5) Break Points:  $\frac{dP}{ds} = \frac{d}{ds} \left( \frac{s}{s^2 + K_0} \right) = \frac{(1)(s^2 + K_0) - s(2s)}{(s^2 + K_0)^2} = \frac{-s^2 + K_0}{(s^2 + K_0)^2} = 0$

$\Rightarrow s^2 - K_0 = 0 \Rightarrow s = \pm\sqrt{K_0}$

$\Rightarrow$  break point on the RL diagram at

$s = -\sqrt{K_0}$

