Intermediate Dynamics
Natural Frequencies and Mode Shapes

To calculate the ***natural frequencies*** and ***mode shapes*** for ***multiple degree-of-freedom*** (DOF), rigid-body systems, the equations of motion (EOM) must first be ***linearized*** about some equilibrium (or steady-state) position and expressed in the following matrix form.

$$\boxed{[M]\{\Delta\ddot{q}\} + [C]\{\Delta\dot{q}\} + [K]\{\Delta q\} = \{F(t)\}}$$

Here, $[M]$, $[C]$, and $[K]$ represent the system's ***mass***, ***damping***, and ***stiffness matrices***, vector $\{\Delta q\}$ represents ***changes*** in all the ***generalized coordinates*** from their equilibrium values, and vector $\{F(t)\}$ represents all the forces acting on the system. Setting the ***damping matrix*** and ***force vector*** to ***zero*** gives an equation of motion representing ***free***, ***undamped response***.

$$\boxed{[M]\{\Delta\ddot{q}\} + [K]\{\Delta q\} = \{0\}} \qquad \text{(\textbf{\textit{free, undamped response}})}$$

Following the pattern for single DOF systems, look for a solution of the following form.

$$\boxed{\{\Delta q\} = e^{j\omega t}\{u\}}$$

This equation describes a ***steady-state***, ***undamped*** solution, with $\{u\}$ representing the ***mode shape*** of the oscillations. Substituting this into the differential EOM gives

$$\boxed{\left([K] - \omega^2[M]\right)\{u\} = \{0\}}$$

The problem now is to find ω^2 and $\{u\}$ that satisfy this ***algebraic*** equation. This is called an ***eigenvalue problem***.

For this equation to have a ***non-zero*** solution for $\{u\}$, the ***determinant*** of the coefficient matrix must be ***zero***. That is,

$$\boxed{\det\left([K] - \omega^2[M]\right) = 0}$$

If the matrices $[M]$ and $[K]$ are $N \times N$ matrices, the solution to this equation yields N values for ω^2, and hence, N values for ω. These are the N ***natural frequencies*** of the system. Associated with each of these frequencies is a ***mode shape*** $\{u\}$. The N ***eigenvalues*** of the system are represented by ω^2, and the N ***eigenvectors*** of the system are represented by the vector $\{u\}$.